Linear Algebra and its Applications 459 (2014) 222–236

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# The principal rank characteristic sequence over various fields

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ARTICLE INFO

*Article history:*
Received 12 September 2013
Accepted 27 June 2014
Available online 24 July 2014
Submitted by R. Brualdi

*MSC:*
15A15
15A03
15A18
15B57
05C50

*Keywords:*
Principal rank characteristic sequence

ABSTRACT

Given an $n \times n$ matrix, its principal rank characteristic sequence is a sequence of length $n+1$ of 0s and 1s where, for $k = 0, 1, \ldots, n$, a 1 in the $k$th position indicates the existence of a principal submatrix of rank $k$ and a 0 indicates the absence of such a submatrix. The principal rank characteristic sequences for symmetric matrices over various fields are investigated, with all such attainable sequences determined for all $n$ over any field with characteristic 2. A complete list of attainable sequences for real symmetric matrices of order 7 is reported.



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[1] Research supported in part by an NSERC Discovery grant.

http://dx.doi.org/10.1016/j.laa.2014.06.045
0024-3795/

Minor
Rank
Symmetric matrix
Hermitian matrix
Finite field

## 1. Introduction

Given an $n \times n$ symmetric matrix $A$ over some field $\mathbb{F}$, the *principal rank characteristic sequence* of $A$ (abbreviated pr-sequence or $\mathrm{pr}(A)$) is defined as $\mathrm{pr}(A) = r_0]r_1r_2 \cdots r_n$ where

$$r_k = \begin{cases} 1 & \text{if } A \text{ has a principal submatrix of rank } k; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $r_0 = 1$ if and only if $A$ has a 0 diagonal entry. Brualdi et al. [1] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [4]; see also [6]. In [1] there is also mention of the case $\mathbb{F} = \mathbb{C}$ and the complex Hermitian matrix case. Note that here we denote a pr-sequence by $r_0]r_1r_2 \cdots r_n$ (rather than by $r_0r_1r_2 \cdots r_n$ as in [1]) to visually emphasize the special nature of $r_0$.

We use the following result to determine the rank, and hence to work with pr-sequences. Here $A[S|T]$ denotes the submatrix of $A$ in rows indexed by the set $S$ and columns indexed by the set $T$. If $S = T$, then we write $A[S]$ for the principal submatrix lying in rows and columns indexed by the set $S$.

**Theorem 1.1.** *If $A \in \mathbb{F}^{n\times n}$ is symmetric, or $A \in \mathbb{C}^{n\times n}$ is Hermitian, then* $\operatorname{rank} A = \max\{|S| : \det(A[S]) \neq 0\}$ *(where the maximum over the empty set is defined to be 0).*

**Proof.** This is immediate from [3, Corollary 8.9.2] for symmetric matrices, and for $A \in \mathbb{C}^{n\times n}$ Hermitian it follows from the equality of algebraic and geometric multiplicity of the eigenvalue zero. □

All matrices in this paper are square, and unless specified otherwise all matrices are symmetric. We are interested in which pr-sequences are *attainable*, i.e., can be attained by some matrix, and also which sequences are *forbidden*, i.e., no matrix attains the sequence. The case $\mathbb{F} = \mathbb{R}$ was studied by Brualdi et al. [1], and in this paper we continue the investigation into pr-sequences by considering the problem over different fields (Sections 2 and 3) and extending the results of [1] over $\mathbb{R}$ (Section 4). In particular, in Section 3 we identify all attainable pr-sequences of all orders over any field with characteristic 2. For some results we use the $(0, 1)$ adjacency matrix of a graph $G$, denoted by $A(G)$, and in Section 5 we give results for pr-sequences of such matrices with full rank.