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The generators of the solution space for a system of inequalities ☆

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The generators of the solution space for a system of inequalities $A \otimes \mathbf{x} \geq \mathbf{x}$ are considered in this paper, where A is a square matrix over $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $\mathbf{x}$ is a column vector and $\otimes$ is a max-plus composition. This paper presents some properties of solutions to the system of inequalities and proposes an algorithm to find a set of generators for the solution space of the system of inequalities with $A \in \overline{\mathbb{R}}^{n \times n}$.

1. Introduction

In max-algebra, $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$ for $a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. The max-plus algebra $(\overline{\mathbb{R}}, \oplus, \otimes)$ is useful for solving a class of nonlinear problems appearing

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for instance in automata theory, scheduling theory, and discrete event systems (see e.g. [6,7,11,14]).

Consider the system in which the machines $P_1, \cdots, P_n$ work in stages. In each stage all machines simultaneously produce components necessary for the next stage of some or all other machines. Let $x_i(r)$ be the starting time of the rth stage on machine P_i ($i = 1, \cdots, n$) and let a_{ij} denote the duration of the operation at which the jth machine prepares a component necessary for the ith machine in the $(r+1)$st stage ($i, j = 1, \cdots, n$). Then

$$x_i(r+1) = \max\{x_1(r) + a_{i1}, \cdots, x_n(r) + a_{in}\}, \quad i = 1, \cdots, n;\ r = 0, 1, \cdots$$

or, in max-algebraic notation

$$\mathbf{x}(r+1) = A \otimes \mathbf{x}(r), \quad r = 0, 1, \cdots$$

where $A = (a_{ij})$ is the production matrix. We say that the system reaches a steady regime (see e.g. [5]) if it eventually moves forward in regular steps, that is, if for some λ and r_0 we have $\mathbf{x}(r+1) = \lambda \otimes \mathbf{x}(r)$ for all $r \geq r_0$. This implies $A \otimes \mathbf{x}(r) = \lambda \otimes \mathbf{x}(r)$ for all $r \geq r_0$. Therefore, the steady regime is reached if and only if for some λ and r, $\mathbf{x}(r)$ is a solution to

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}.$$

Throughout the paper, for positive integers n, m the symbol $\overline{\mathbb{R}}^{n\times m}$ will denote the set of all $n \times m$ matrices over $\overline{\mathbb{R}}$, and ε is used instead of $-\infty$ and for convenience we also denote by the same symbol any vector or matrix whose every component is ε. Given $A \in \overline{\mathbb{R}}^{n\times n}$, the problem of finding a vector $\mathbf{x} \in \overline{\mathbb{R}}^n = \overline{\mathbb{R}}^{n\times 1}$, $\mathbf{x} \neq \varepsilon$ and a scalar $\lambda \in \overline{\mathbb{R}}$ satisfying

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}$$

is called the (max-algebraic) eigenproblem, the vector $\mathbf{x}$ is called an eigenvector and the scalar λ is called an eigenvalue of A.

The eigenproblem is of key importance in max-algebra. It has been studied in research papers from the early 1960's [5]. Full solutions of the eigenproblem in the case of irreducible matrices have been presented in [6], see also [1,8,10,12]. The Spectral Theorem for reducible matrices has been proved in [2] and [9].

Let $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \mathbb{R}$. A vector $\mathbf{x} \in \overline{\mathbb{R}}^n$, $\mathbf{x} \neq \varepsilon$, satisfying $A \otimes \mathbf{x} \leq \lambda \otimes \mathbf{x}$ is called a subeigenvector of A with associated eigenvalue λ and denote

$$V_*(A, \lambda) = \{\mathbf{x} \in \overline{\mathbb{R}}^n : A \otimes \mathbf{x} \leq \lambda \otimes \mathbf{x}, \mathbf{x} \neq \varepsilon\}.$$