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Upper bounds on the (signless) Laplacian eigenvalues of graphs

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ABSTRACT

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Also let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$ be the Laplacian eigenvalues and signless Laplacian eigenvalues of G, respectively. In this paper we obtain $\mu_i(G) \leq i - 1 + \min_{U_i} \max\{|N_H(v_k) \cup N_H(v_j)|\colon v_k v_j \in E(H)\}$, where $N_H(v_k)$ is the set of neighbors of vertex v_k in $V(H) = V(G) \backslash U_i$, U_i is any $(i-1)$-subset of $V(G)$ (here, we agree that $i \in \{1, \ldots, n-1\}$ and $\mu_i(G) \leq i - 1$ if $E(H) = \emptyset$). For any graph G, this bound does not exceed the order of G. Moreover, we prove that

$$\max\{\mu_i(G), q_i(G)\} \leq \max_{i \leq k \leq n} \left\{ d_G(v_k) + \sum_{v_j \in N_G(v_k) \cap N} \frac{d_G(v_j)}{d_G(v_k)} \right\} \leq 2 d_G(v_i),$$

where $d_G(v_i)$ is the i-th largest degree of G and $N = \{v_i, v_{i+1}, \ldots, v_n\}$.

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1. Introduction

Throughout this paper, let $G = (V, E)$ be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Also let $\overline{G}$ be the complement graph of G. For each $v_i \in V(G)$, the set of neighbors of vertex v_i is denoted by $N_G(v_i)$. Let $d_G(v_i)$ be the degree of vertex v_i $(i = 1, 2, \ldots, n)$. If there is no confusion, sometimes we write d_i in place of $d_G(v_i)$. The diameter of G is the maximum distance between any two vertices of G. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the $(0,1)$-adjacency matrix of G. It is well known that all the Laplacian eigenvalues of $L(G)$ are non-negative. Throughout this paper let

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$$

be the eigenvalues of $L(G)$. The *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. It is easy to see that $Q(G)$ is also positive semidefinite [3] and hence its eigenvalues can be arranged as

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0.$$

There exists a vast literature that studies the Laplacian eigenvalues and their relationship with various properties of graphs. We refer [1,4,15] to the reader for surveys and more information. Most of the effort in the study of Laplacian eigenvalues has naturally been concentrated on the extremal non-trivial eigenvalues [4,8]. The integer Laplacian eigenvalues as well as real eigenvalues are studied in [6]. Gutman et al. [10] recently discovered connection between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of the underlying molecular graphs. So it is significant and necessary to investigate the relations between the graph theoretic properties of G and its eigenvalues. Till now, plenty of upper bounds on the largest Laplacian eigenvalue of graphs [4, Section 1] have been given. Only Li and Pan [13] and Das [5,7] presented lower bounds on the second largest Laplacian eigenvalue of graphs and trees. Guo [9] gave an upper bound on the second largest Laplacian eigenvalue of trees with perfect matching. Brouwer and Haemers [2] proved a conjecture of Guo, which is on the lower bound of the i-th largest Laplacian eigenvalues of connected graph.

The paper is organized in the following way. In Section 2, we list some well known results related to eigenvalues of the corresponding Laplacian matrix of a graph and symmetric matrix. In Section 3, we obtain the following upper bound on $\mu_i(G)$,

$$\mu_i(G) \leq i - 1 + \min_{U_i} \max\bigl\{\bigl|N_H(v_k) \cup N_H(v_j)\bigr|:\ v_k v_j \in E(H)\bigr\},$$

where $N_H(v_k)$ is the set of neighbors of vertex v_k in $V(H) = V(G) \backslash U_i$, U_i is any $(i-1)$-subset of $V(G)$ (here and in Theorem 3.1, we agree that $i \in \{1, \ldots, n-1\}$