Linear Algebra and its Applications 459 (2014) 357–367

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The point, residual and continuous spectrum of an upper triangular operator matrix ☆

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A R T I C L E   I N F O

Article history:
Received 8 January 2014
Accepted 8 July 2014
Available online 31 July 2014
Submitted by Y. Wei

MSC:
47A10
47A53
47A55

Keywords:
Operator matrix
Invertibility
Residual spectrum
Continuous spectrum
Completion

A B S T R A C T

In this paper, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we completely describe the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is injective, $\mathcal{R}(M_C)$ is not dense in $\mathcal{H} \oplus \mathcal{K}$, respectively and describe residual and continuous spectrum of the upper triangular operator matrix M_C.

1. Introduction

Let $\mathcal{H}, \mathcal{K}$ be separable Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H}, \mathcal{H})$ as $\mathcal{B}(\mathcal{H})$. For a given

☆ Supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia.

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http://dx.doi.org/10.1016/j.laa.2014.07.010
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$A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. Let $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \operatorname{codim} \mathcal{R}(A)$.

The spectrum $\sigma(A)$ is split into the point spectrum $\sigma_p(A)$, the residual spectrum $\sigma_r(A)$ and the continuous spectrum $\sigma_c(A)$ which are defined by

$$\sigma_p(A) = \{\lambda \in \mathbb{C}\colon\ \mathcal{N}(A - \lambda I) \neq \{0\}\},$$
$$\sigma_r(A) = \{\lambda \in \mathbb{C}\colon\ \mathcal{N}(A - \lambda I) = \{0\},\ \overline{\mathcal{R}(A - \lambda I)} \neq \mathcal{H}\},$$
$$\sigma_c(A) = \{\lambda \in \mathbb{C}\colon\ \mathcal{N}(A - \lambda I) = \{0\},\ \mathcal{R}(A - \lambda I) \neq \overline{\mathcal{R}(A - \lambda I)} = \mathcal{H}\}.$$

It is easy to conclude (see [8, p. 92]) that $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ are pairwise disjoint and

$$\sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A) = \sigma(A).$$

There are many papers which consider some types of invertibility and regularity of upper-triangular operator matrices

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}$$

(see [1–9] and references therein), as well as various types of spectra of M_C. In particular, the continuous, point and residual spectra of M_C were considered in [5,9]. In this paper we approach the problem using a technique different than those employed in these papers and, in addition, for given operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we completely describe the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is injective, $\mathcal{R}(M_C)$ is not dense in $\mathcal{H} \oplus \mathcal{K}$, $\lambda \in \sigma_r(M_C)$ and $\lambda \in \sigma_c(M_C)$, for some scalar λ, respectively.

Notice that for given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the set of all $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that M_C is injective, $\mathcal{R}(M_C)$ is not dense in $\mathcal{H} \oplus \mathcal{K}$, $0 \in \sigma_r(M_C)$ and $0 \in \sigma_c(M_C)$ will be denoted by $S_I(A, B)$, $S_{ND}(A, B)$, $S_R(A, B)$, $S_C(A, B)$, respectively.

2. The point, residual and continuous spectrum of an operator matrix M_C

In [4] and [6], it is proved that for $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the operator matrix M_C is invertible for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if

(i) A is left invertible,
(ii) B is right invertible,
(iii) $\mathcal{N}(B) \cong X/R(A)$.