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On tropical fractional linear programming

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ABSTRACT

Very recently, tropical counterparts of fractional linear programs have been studied. Some algorithms were proposed for solving them, with techniques ranging from bisection methods to homeomorphisms to formal power series. In this paper, some algorithms are also proposed. They mainly rely in the ability of finding the greatest and smallest solutions of tropical equations, a subject that was discussed in a previous work of the authors [13].

1. Introduction

Tropical Fractional Linear Programs (denoted as **TFLP** hereafter) are problems of the form [11] (see Section 2 for the notations)

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$$\begin{aligned} &\max / \min \quad \left(\mathbf{w}^T \mathbf{p} \oplus \alpha\right) \oslash \left(\mathbf{f}^T \mathbf{p} \oplus \beta\right) \\ &\text{such that} \quad R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}. \end{aligned} \tag{1}$$

This formulation can be used to compute the tightest inequality of the form $p_i - p_j \geq K$ if $\mathbf{p}$ is inside a tropical polyhedron, which finds applications in static analysis (see [11]). It can also be used to check if a set of equalities $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$ implies another equality $\mathbf{w}^T \mathbf{p} \oplus \alpha = \mathbf{f}^T \mathbf{p} \oplus \beta$ without the burden of finding all solutions explicitly.[1] This is true if and only if both max and min versions of Problem 1 have optimal value 0. This is worthy of mentioning because, in contrast with the traditional algebra, in the tropical setting there are equalities that can be logically deduced from a set of other equalities, but *cannot* be obtained by taking tropical linear combinations of these (see [10]). Hence, it is not always possible to expect to claim that $R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s}$ does not imply $\mathbf{w}^T \mathbf{p} \oplus \alpha = \mathbf{f}^T \mathbf{p} \oplus \beta$ by verifying the solvability of the equations $\mathbf{z}^T R = \mathbf{w}$, $\mathbf{z}^T \mathbf{r} = \alpha$, $\mathbf{z}^T S = \mathbf{f}^T$ and $\mathbf{z}^T \mathbf{s} = \beta$ for $\mathbf{z}$.[2] With an efficient algorithm for solving TFLPs, one can check the validity of this proposition in an easy manner.

These kind of optimization problems have begun receiving attention from scientific community not a long time ago. By the authors' knowledge, the first published work that has solved a particular case of Problem 1 (save the very particular cases in which it can be solved by a direct application of residuation theory, such as $A\mathbf{x} \preceq \mathbf{b}$) was [4]. This special case is when $\mathbf{f} = \perp$ and $\beta = 0$ (Tropical Linear Programs, denoted as **TLP** hereafter)

$$\begin{aligned} &\max / \min \quad \mathbf{w}^T \mathbf{p} \oplus \alpha \\ &\text{such that} \quad R\mathbf{p} \oplus \mathbf{r} = S\mathbf{p} \oplus \mathbf{s} \end{aligned} \tag{2}$$

and an algorithm was presented to solve them. This formulation can be used to solve optimization problems for multiprocessor systems (see [4]). The idea is that it is possible to check whether a value of objective function in Problem 1 is achievable by solving a tropical affine equation. Thus, if a lower and an upper bounds for the objective function are derived, it is possible to use a bisection method to search for the optimal value. The recent paper [5] pursues an integer solution to the problem when the entries are real numbers, also using a similar bisection approach.

Gaubert et al. [11] studied the complete problem, and derived a Newton-like algorithm which works by solving a sequence of mean-payoff games. More recently, Allamigeon et al. [1] used the field of *generalized Puiseux series* over $\mathbb{R}$, $\mathbb{K}$ (formal power series in one variable in which the exponents can be any real number) to develop an alternative approach to the problem. It explores the idea of *valuation*, a function which maps each Puiseux series to the opposite of its smallest exponent with a non-zero coefficient. In a

[1] Finding all solutions of a tropical linear equation can be a very time consuming task, so, it is advantageous to avoid it whenever possible.

[2] The analogue of this affirmation for traditional algebra, i.e. $A\mathbf{x} = \mathbf{b}$ implies $\mathbf{c}^T \mathbf{x} = d$ only if (the "if" part is trivial) there exists $\mathbf{y}$ such that $\mathbf{y}^T A = \mathbf{c}^T$ and $\mathbf{y}^T \mathbf{b} = d$, is a consequence of the Farkas lemma.