Linear Algebra and its Applications 459 (2014) 465–474

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On vector configurations that can be realized in the cone of positive matrices

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ARTICLE INFO

Article history:
Received 13 April 2010
Accepted 15 July 2014
Available online 7 August 2014
Submitted by P. Semrl

MSC:
15B48

Keywords:
Vector configuration
Cone of positive semidefinite matrices
Von Neumann algebra
Trace

ABSTRACT

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in an inner product space. Can we find a dimension d and positive (semidefinite) matrices $A_1, \ldots, A_n \in M_d(\mathbb{C})$ such that $\operatorname{Tr}(A_k A_l) = \langle \mathbf{v}_k, \mathbf{v}_l \rangle$ for all $k, l = 1, \ldots, n$? For such matrices to exist, one must have $\langle \mathbf{v}_k, \mathbf{v}_l \rangle \geq 0$ for all $k, l = 1, \ldots, n$. We prove that if $n < 5$ then this trivial necessary condition is also a sufficient one and find an appropriate example showing that from $n = 5$ this is not so – even if we allowed realizations by positive operators in a von Neumann algebra with a faithful normal tracial state.

The fact that the first such example occurs at $n = 5$ is similar to what one has in the well-investigated problem of *positive factorization* of positive (semidefinite) matrices. If the matrix $(\langle \mathbf{v}_k, \mathbf{v}_l \rangle)_{(k,l)}$ has a positive factorization, then matrices $A_1, \ldots, A_n$ as above exist. However, as we show by a large class of examples constructed with the help of the Clifford algebra, the converse implication is false.

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http://dx.doi.org/10.1016/j.laa.2014.07.017
0024-3795/

1. Introduction

1.1. Motivation

Throughout this paper, the term “positive matrix” will mean “positive semidefinite matrix”. The aim of the paper is to study a geometrical property of the cone $\mathcal{C}_d$ of positive matrices in $M_d(\mathbb{C})$ in the “large dimensional limit”: we investigate if a given configuration of vectors can be embedded in $\mathcal{C}_d$ for *some* (possibly very large) $d \in \mathbb{N}$. Note that for $d_1 \le d_2$ we have $\mathcal{C}_{d_1} \hookrightarrow \mathcal{C}_{d_2}$ in a natural manner, so if a configuration can be embedded in $\mathcal{C}_{d_1}$, then it can be embedded in $\mathcal{C}_{d_2}$.

To explain the precise meaning of our question, consider the real vector space formed by the self-adjoint elements of $M_d(\mathbb{C})$. It has a natural inner product defined by the formula

$$\langle A, B\rangle \equiv \frac{1}{d}\operatorname{Tr}(AB) \quad \bigl(A^* = A,\ B^* = B \in M_d(\mathbb{C})\bigr), \tag{1}$$

making it a *Euclidean* space. Our cone $\mathcal{C}_d$ is a convex cone in this space with a “sharp end-point” at zero[1]: if $A, B \in \mathcal{C}_d$ then $\langle A, B\rangle \ge 0$.

Suppose we are given n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in a Euclidean space. Embedding them in an inner product preserving way in $\mathcal{C}_d$ means finding n positive matrices $A_1, \dots, A_n \in \mathcal{C}_d$ such that

$$\langle \mathbf{v}_j, \mathbf{v}_k\rangle = \langle A_j, A_k\rangle \equiv \frac{1}{d}\operatorname{Tr}(A_j A_k). \tag{2}$$

Since, as was mentioned, the angle between any two vectors in $\mathcal{C}_d$ is $\le \pi/2$, one can only hope to embed these vectors if $\langle \mathbf{v}_j, \mathbf{v}_k\rangle \ge 0$ for all $j, k = 1, \dots, n$. So suppose this condition is satisfied. Does it follow that the given vectors can be embedded in $\mathcal{C}_d$ for some (possibly very large) d? If not, can we characterize the configurations that can be embedded? To our knowledge, these questions have not been considered in the literature.

We postpone the summary of our results to the next subsection and note that there is a well-investigated problem – namely the problem of *positive factorization* – which has some relation to our questions. The relation between the two topics will also be discussed in the next subsection; here we shall only explain our original motivation.

If A is positive and $A \ne 0$, then $\operatorname{Tr}(A) > 0$, so the affine hyperplane $\{X : \operatorname{Tr}(X) = 1\}$ intersects each ray of the cone $\mathcal{C}_d$ exactly once and geometric properties of this cone can be equally well studied by considering just the intersection

$$\mathcal{S}_d = \bigl\{A \in M_d(\mathbb{C}) : A \ge 0,\ \operatorname{Tr}(A) = 1\bigr\}. \tag{3}$$

[1] Actually one has the much stronger property that an element X in this space belongs to the cone $\mathcal{C}_d$ if and only if $\langle X, A\rangle \ge 0$ for all $A \in \mathcal{C}_d$; i.e. $\mathcal{C}_d$ is a *self-dual* cone.