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On semi-convergence of generalized skew-Hermitian triangular splitting iteration methods for singular saddle-point problems

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ABSTRACT

Recently, Krukier et al. (2014) [13] proposed an efficient *generalized skew-Hermitian triangular splitting* (GSTS) iteration method for nonsingular saddle-point linear systems with strong skew-Hermitian parts. In this work, we further use the GSTS method to solve *singular* saddle-point problems. The semi-convergence properties of GSTS method are analyzed by using singular value decomposition and Moore–Penrose inverse, under suitable restrictions on the involved iteration parameters. Numerical results are presented to demonstrate the feasibility and efficiency of the GSTS iteration methods, both used as solvers and preconditioners for GMRES method.

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1. Introduction

Consider the following saddle-point linear system:

$$\mathcal{A}u \equiv \begin{pmatrix} M & E \\ -E^* & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv f, \tag{1.1}$$

where $M \in \mathbb{C}^{p\times p}$ is an Hermitian positive definite matrix, $E \in \mathbb{C}^{p\times q}$ is a rectangular matrix satisfying $q \le p$, and $f \in \mathbb{C}^{p+q}$ is a given vector in the range of $\mathcal{A} \in \mathbb{C}^{(p+q)\times(p+q)}$, with $f_1 \in \mathbb{C}^p$ and $f_2 \in \mathbb{C}^q$. This kind of linear systems arise in a variety of scientific and engineering applications, such as computational fluid dynamics, constrained optimization, optimal control, weighted least-squares problems, electronic networks, computer graphics, etc., and typically result from mixed or hybrid finite element approximation of second-order elliptic problems or the Stokes equations; see [1–5].

When matrix E is of full column rank, the saddle-point matrix $\mathcal{A}$ is nonsingular. A number of effective iteration methods, such as matrix splitting iteration methods, minimum residual methods, Krylov subspace iteration methods, etc., have been proposed in the literature to approximate the unique solution of the nonsingular saddle-point problems (1.1); see [6–12] and the references therein.

Split matrix $\mathcal{A}$ into its Hermitian and skew-Hermitian parts as $\mathcal{A} = \mathcal{A}_H + \mathcal{A}_S$, where

$$\mathcal{A}_H = \frac{1}{2}(\mathcal{A} + \mathcal{A}^*) = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_S = \frac{1}{2}(\mathcal{A} - \mathcal{A}^*) = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}. \tag{1.2}$$

Let $\mathcal{K}_L$ and $\mathcal{K}_U$ be, respectively, the strictly lower-triangular and the strictly upper-triangular parts of $\mathcal{A}_S$ satisfying

$$\mathcal{A}_S = \mathcal{K}_L + \mathcal{K}_U = \begin{pmatrix} 0 & 0 \\ -E^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}. \tag{1.3}$$

Recently, on the basis of the above two splittings, Krukier et al. [13] proposed a generalized skew-Hermitian triangular splitting (GSTS) iteration method for solving the linear systems with strong skew-Hermitian parts. The iteration scheme of the GSTS method is

$$u^{(k+1)} = u^{(k)} - \tau\mathcal{B}(\omega_1,\omega_2)^{-1}\big(\mathcal{A}u^{(k)} - f\big), \tag{1.4}$$

where τ is a positive parameter and

$$\mathcal{B}(\omega_1,\omega_2) = (\mathcal{B}_c + \omega_1\mathcal{K}_L)\mathcal{B}_c^{-1}(\mathcal{B}_c + \omega_2\mathcal{K}_U). \tag{1.5}$$