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Cauchy–Binet for pseudo-determinants

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ABSTRACT

The pseudo-determinant $\mathrm{Det}(A)$ of a square matrix A is defined as the product of the nonzero eigenvalues of A. It is a basis-independent number which is up to a sign the first nonzero entry of the characteristic polynomial of A. We prove $\mathrm{Det}(F^T G) = \sum_P \det(F_P)\det(G_P)$ for any two $n \times m$ matrices F, G. The sum to the right runs over all $k \times k$ minors of A, where k is determined by F and G. If $F = G$ is the incidence matrix of a graph this directly implies the Kirchhoff tree theorem as $L = F^T G$ is then the Laplacian and $\det^2(F_P) \in \{0, 1\}$ is equal to 1 if P is a rooted spanning tree. A consequence is the following Pythagorean theorem: for any self-adjoint matrix A of rank k, one has $\mathrm{Det}^2(A) = \sum_P \det^2(A_P)$, where $\det(A_P)$ runs over $k \times k$ minors of A. More generally, we prove the polynomial identity $\det(1 + xF^T G) = \sum_P x^{|P|}\det(F_P)\det(G_P)$ for classical determinants det, which holds for any two $n \times m$ matrices F, G and where the sum on the right is taken over all minors P, understanding the sum to be 1 if $|P| = 0$. It implies the Pythagorean identity $\det(1 + F^T F) = \sum_P \det^2(F_P)$ which holds for any $n \times m$ matrix F and sums again over all minors F_P. If applied to the incidence matrix F of a finite simple graph, it produces the Chebotarev–Shamis forest theorem telling that $\det(1 + L)$ is the number of rooted spanning forests in the graph with Laplacian L.

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1. Introduction

The Cauchy–Binet theorem for two $n \times m$ matrices F, G with $n \geq m$ tells that

$$\det\left(F^T G\right) = \sum_P \det(F_P)\det(G_P), \tag{1}$$

where the sum is over all $m \times m$ square sub-matrices P and F_P is the matrix F masked by the pattern P. In other words, F_P is an $m \times m$ matrix obtained by deleting $n - m$ rows in F and $\det(F_P)$ is a minor of F. In the special case $m = n$, the formula is the product formula $\det(F^T G) = \det(F^T)\det(G)$ for determinants. Direct proofs can be found in [39,34,41]. An elegant multi-linear proof is given in [22], where it is called "almost tautological". A graph-theoretical proof using the Lindström–Gessel–Viennot lemma sees matrix multiplication as a concatenation process of directed graphs and determinants as a sum of weighted path integrals [3]. The classical Cauchy–Binet theorem implies the Pythagorean identity $\det(F^T F) = \sum_P \det^2(F_P)$ for square matrices which is also called Lagrange Identity [4], where P runs over all $m \times m$ sub-matrices of F. This formula is used in multivariable calculus in the form $|\vec{v}|^2|\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 = |\vec{v} \wedge \vec{w}|^2$ and is useful to count the number of basis choices in matroids [2]. The Cauchy–Binet formula assures that the determinant is compatible with the matrix product. Historically, after Leibniz introduced determinants in 1693 [40], and Vandermonde made it into a theory in 1776 [24], Binet and Cauchy independently found the product formula for the determinant around 1812 [8,14,45,9,12], even before matrix multiplication had been formalized. [16,24] noticed that Lagrange had mentioned a similar result before in the three dimensional case. The term "matrix" was used by Sylvester first in 1850 [44,55,56]. The book [24] mentions that Binet's proof was not complete. It was Cayley who looked first at the matrix algebra [15,32,42,43]. It is evident today that the Cauchy–Binet formula played a pivotal role for the development of matrix algebra. Early textbooks like [50] illustrate how much the notation of determinants has changed over time.

In this paper, we extend the Cauchy–Binet formula (1) to matrices with determinant 0. Theorem 2 will follow from Theorem 7 and imply new formulas for classical determinants like Theorem 8. The pseudo-determinant $\mathrm{Det}(A)$ for a square matrix A is defined as the product of the nonzero eigenvalues of A, with the assumption $\mathrm{Det}(0) = 1$ for a matrix with all zero eigenvalues like $A = 0$ or nilpotent matrices. The later assumption renders all formulas also true for zero matrices. Looking at singular matrices with pseudo-determinants opens a new world, which formula (1) buries under the trivial identity "$0 = 0$". The extension of Cauchy–Binet to pseudo-determinants is fascinating because these determinants are not much explored and because Cauchy–Binet for pseudo-determinants is not a trivial extension of the classical theorem. One reason is that the most commonly used form of Cauchy–Binet is false, even for diagonal square matrices A, B: while $\mathrm{Det}(AB) = \mathrm{Det}(BA)$ is always true, we have in general: