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The algebra $U_q(\mathfrak{sl}_2)$ in disguise

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ABSTRACT

We discuss a connection between the algebra $U_q(\mathfrak{sl}_2)$ and the tridiagonal pairs of q-Racah type. To describe the connection, let x, $y^{\pm 1}$, z denote the equitable generators for $U_q(\mathfrak{sl}_2)$. Let $U_q^\vee$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by x, y^{-1}, z. Using a tridiagonal pair of q-Racah type we construct two finite-dimensional $U_q^\vee$-modules. The constructions yield two nonstandard presentations of $U_q^\vee$ by generators and relations. These presentations are investigated in detail.

1. Introduction

This paper is about a connection between the algebra $U_q(\mathfrak{sl}_2)$ and a linear-algebraic object called a tridiagonal pair [7]. To describe the connection, we briefly recall the algebra $U_q(\mathfrak{sl}_2)$ [12,13]. We will use the equitable presentation, which was introduced in [11]. Given a field $\mathbb{F}$, the $\mathbb{F}$-algebra $U_q(\mathfrak{sl}_2)$ has generators x, $y^{\pm 1}$, z and relations $yy^{-1} = y^{-1}y = 1$,

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$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

Let $U_q^\vee$ denote the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by x, y^{-1}, z. We now briefly turn our attention to tridiagonal pairs. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Roughly speaking, a tridiagonal pair on V is a pair of diagonalizable $\mathbb{F}$-linear maps on V, each acting on the eigenspaces of the other one in a block-tridiagonal fashion. There is a general family of tridiagonal pairs said to have q-Racah type [8, Definition 3.1]. Let $\mathbf{A}$, $\mathbf{A}^*$ denote a tridiagonal pair on V that has q-Racah type. Associated with this tridiagonal pair are two additional $\mathbb{F}$-linear maps $K : V \to V$ [4, Definition 3.1] and $B : V \to V$ [4, Definition 3.2], which are roughly described as follows. The maps K, B are diagonalizable, and their eigenspace decompositions are among the split decompositions of V with respect to $\mathbf{A}$, $\mathbf{A}^*$ [7, Section 4]. According to [4, Lemma 3.6, Theorem 9.9], there exists $a \in \mathbb{F}\backslash\{0, 1, -1\}$ such that

$$\frac{qK\mathbf{A} - q^{-1}\mathbf{A}K}{q - q^{-1}} = aK^2 + a^{-1}I, \qquad \frac{qB\mathbf{A} - q^{-1}\mathbf{A}B}{q - q^{-1}} = a^{-1}B^2 + aI,$$

$$aK^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}}KB - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}}BK + a^{-1}B^2 = 0.$$

The main purpose of this paper is to explain what the above three equations have to do with $U_q(\mathfrak{sl}_2)$. To summarize the answer, consider an $\mathbb{F}$-algebra defined by generators and relations; the generators are symbols K, B, $\mathbf{A}$ and the relations are the above three equations. We are going to show that this algebra is isomorphic to $U_q^\vee$. The existence of this isomorphism yields a presentation of $U_q^\vee$ by generators and relations. There is another presentation of $U_q^\vee$ with similar features, obtained by inverting q, a, K, B. In our main results, we describe these two presentations in a comprehensive way that places them in a wider context. We will summarize our results shortly.

In the theory of $U_q(\mathfrak{sl}_2)$, it is convenient to introduce elements ν_x, ν_y, ν_z defined by

$$\nu_x = q(1 - yz), \qquad \nu_y = q(1 - zx), \qquad \nu_z = q(1 - xy).$$

One significance of ν_x, ν_y, ν_z is that

$$x\nu_y = q^2\nu_y x, \qquad x\nu_z = q^{-2}\nu_z x,$$
$$y\nu_z = q^2\nu_z y, \qquad y\nu_x = q^{-2}\nu_x y,$$
$$z\nu_x = q^2\nu_x z, \qquad z\nu_y = q^{-2}\nu_y z.$$

We now summarize our results. Fix a nonzero $a \in \mathbb{F}$ such that $a^2 \neq 1$. Define elements X, Z of $U_q(\mathfrak{sl}_2)$ by

$$X = a^{-2}x + \left(1 - a^{-2}\right)y^{-1}, \qquad Z = a^2 z + \left(1 - a^2\right)y^{-1}.$$