Linear Algebra and its Applications 459 (2014) 586–594

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# Jacobi–Jordan algebras

Dietrich Burde [a], Alice Fialowski [b,*]

[a] *Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria*
[b] *Eötvös Loránd University, Institute of Mathematics, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary*

A R T I C L E   I N F O

*Article history:*
Received 6 July 2014
Accepted 22 July 2014
Available online 7 August 2014
Submitted by V.V. Sergeichuk

*MSC:*
primary 17C10, 17C55

*Keywords:*
Jacobi identity
Jordan algebra
Commutative nilalgebra

A B S T R A C T

We study finite-dimensional commutative algebras, which satisfy the Jacobi identity. Such algebras are Jordan algebras. We describe some of their properties and give a classification in dimensions $n < 7$ over algebraically closed fields of characteristic not 2 or 3.


## 1. Introduction

Finite-dimensional commutative associative algebras (with and without 1) have been studied intensively. In particular, many different classification results have appeared in the literature, see [15,7,10,11] for recent publications, and the references given therein. On the other hand, many important classes of commutative *non-associative* algebras have been studied, too. Examples are Jordan algebras, commutative power-associative algebras, commutative alternative algebras, Bernstein algebras and many more. We mention

* Corresponding author.
*E-mail addresses:* dietrich.burde@univie.ac.at (D. Burde), fialowsk@cs.elte.hu (A. Fialowski).

http://dx.doi.org/10.1016/j.laa.2014.07.034

also pre-Lie algebras, Novikov algebras and LR-algebras, which we studied in a geometric context, see [3,2,4]. In each case there, the subclass of commutative algebras therein automatically satisfies associativity. In particular, the classification of such algebras includes the classification of commutative associative algebras as a special case. See for example the classification of Novikov algebras in low dimension [5].

Here we want to study another class of commutative non-associative algebras. They satisfy the Jacobi identity instead of associativity. It turns out that they are a special class of Jordan nilalgebras. For this reason we want to call them *Jacobi–Jordan algebras*. These algebras are also related to Bernstein–Jordan algebras. Although they seem to be similar to Lie algebras at first sight, these algebras are quite different.

We are grateful to Manfred Hartl, who asked what is known about commutative algebras satisfying the Jacobi identity. We are also grateful to Ivan Shestakov for answering some questions on the subject, and paying our attention to Bernstein–Jacobi algebras.

## 2. Jacobi–Jordan algebras

Let $A$ be a finite-dimensional algebra over a field $K$. We will always assume that the characteristic of $K$ is different from 2 or 3, although some of our results also hold for characteristic 2 or 3. We denote the algebra product by $x \cdot y$. The principal powers of an element $x \in A$ are defined recursively by $x^1 = x$ and $x^{i+1} = x \cdot x^i$. Also, let $A^1 = A$ and define $A^k = \sum_{i+j=k} A^i A^j$ recursively for all integers $k \geq 2$. We say that $A$ is *nilpotent*, if there is a positive integer $k$ with $A^k = (0)$. The algebra $A$ is called a *nilalgebra* of nilindex $n$, if the subalgebra generated by any given $a \in A$ is nilpotent, i.e., if $y^n = 0$ for all $y \in A$, and there exists an element $x \in A$ with $x^{n-1} \neq 0$. If an algebra $A$ is a commutative nilalgebra of nilindex $n \leq 2$, then all products vanish because of $x \cdot y = \frac{1}{2}((x+y)^2 - x^2 - y^2) = 0$ for all $x, y \in A$.

**Definition 2.1.** An algebra $A$ over a field $K$ is called a *Jacobi–Jordan* algebra if it satisfies the following two identities,

$$x \cdot y - y \cdot x = 0 \tag{1}$$

$$x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0 \tag{2}$$

for all $x, y, z \in A$.

In other words, Jacobi–Jordan algebras are commutative algebras which satisfy the Jacobi identity. In the case of Lie algebras, the product is not commutative but anticommutative. Then the Jacobi identity has equivalent formulations which are not necessarily equivalent in the commutative case. Obviously the Jacobi identity implies that $x^3 = 0$ for all $x \in A$, since $3 \neq 0$. Hence a Jacobi–Jordan algebra is a nilalgebra. Since the algebra is commutative, we have $x^3 = x \cdot (x \cdot x) = (x \cdot x) \cdot x$.