Linear Algebra and its Applications 452 (2014) 46–67

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# A characterization of bipartite Leonard pairs using the notion of a tail

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ARTICLE INFO

*Article history:*
Received 18 August 2013
Accepted 6 March 2014
Available online 9 April 2014
Submitted by R. Brualdi

*MSC:*
primary 15A21
secondary 05E30

*Keywords:*
Leonard pair
Tridiagonal pair
Distance-regular graph
Bipartite
Orthogonal polynomials

ABSTRACT

Let $V$ denote a vector space with finite positive dimension. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below.

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.
(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

We call such a pair a *Leonard pair* on $V$. Very roughly speaking, a Leonard pair is a linear algebraic abstraction of a $Q$-polynomial distance-regular graph. There is a well-known class of distance-regular graphs said to be bipartite and there is a related notion of a bipartite Leonard pair. Recently, M.S. Lang introduced the notion of a tail for bipartite distance-regular graphs and there is an abstract version of this tail notion. Lang characterized the bipartite $Q$-polynomial distance-regular graphs using tails. In this paper, we obtain a similar characterization of the bipartite Leonard pairs using tails. Whereas Lang's arguments relied on the combinatorics of a distance-regular graph, our results are purely algebraic in nature.

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http://dx.doi.org/10.1016/j.laa.2014.03.010

## 1. Introduction

We begin by recalling the notion of a Leonard pair [9,10]. We will use the following terms. Let $X$ denote a square matrix. Then $X$ is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume $X$ is tridiagonal. Then $X$ is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper, $\mathbb{K}$ will denote a field.

**Definition 1.1.** (See [10, Definition 1.1].) Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on $V$, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below.

- (i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.
- (ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

**Note 1.2.** It is a common notational convention to use $A^*$ to represent the conjugate-transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$, the linear transformations $A$ and $A^*$ are arbitrary subject to (i), (ii) above.

Very roughly speaking, a Leonard pair is a linear algebraic abstraction of a $Q$-polynomial distance-regular graph [1, p. 260], [9, Definition 2.3]. In the theory of distance-regular graphs, there is a well-known set of parameters $a_i$ called intersection numbers. There also exists an abstract version of the $a_i$. A distance-regular graph is said to be bipartite whenever each $a_i$ equals zero. A bipartite Leonard pair is similarly defined.

In [2,3], we extended existing characterizations of $Q$-polynomial distance-regular graphs to obtain characterizations of Leonard pairs. We mention some details. In [5], M.S. Lang introduced the notion of a tail for bipartite distance-regular graphs. In [4], the tail notion was applied to general distance-regular graphs. In [4, Theorem 1.1], these tails were used to characterize $Q$-polynomial distance-regular graphs. In [2, Definition 4.5], we introduced an abstract version of the tail notion and in [2, Theorem 5.1], we used it to characterize Leonard pairs. In [8, Theorem 1.2], the $a_i$ were used to characterize $Q$-polynomial distance-regular graphs. In [3], we used the $a_i$ to characterize Leonard pairs; our main result was [3, Theorem 5.1].

We now summarize the present paper. Our point of departure is Lang's work concerning tails and bipartite distance-regular graphs [5]. In [5, Theorem 1.1], Lang used tails