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Lowest-rank solutions of continuous and discrete Lyapunov equations over symmetric cone

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ABSTRACT

The low-rank solutions of continuous and discrete Lyapunov equations are of great importance but generally difficult to compute in control system analysis and design. Fortunately, Mesbahi and Papavassilopoulos (1997) [30] showed that with the semidefinite cone constraint, the lowest-rank solutions of the discrete Lyapunov inequality can be efficiently solved by a linear semidefinite programming. In this paper, we further show that the lowest-rank solutions of both the continuous and discrete Lyapunov equations over symmetric cone are unique and can be exactly solved by their convex relaxations, the symmetric cone linear programming problems. Therefore, they are polynomial-time solvable. Since the underlying symmetric cone is a more general algebraic setting which contains the semidefinite cone as a special case, our results also answer an open question proposed by Recht, Fazel and Parrilo (2010) [36].

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1. Introduction

The continuous and discrete Lyapunov equations are fundamental matrix equations, whose algebraic equations take the following forms

$$AX + XA^\top = BB^\top, \qquad X - MXM^\top = HH^\top$$

where A, M, B, H and X are matrices of some appropriate dimensions. These equations play a significant role in control theory, model reduction and stochastic analysis of dynamical systems [1,2,4,14]. The theoretical analysis and numerical solutions for these equations have been the topics of numerous publications, see [3,19,24,34,43] and the references therein. Among all solutions, the low-rank ones are of great importance in control system analysis and design. Here the concept of low-rank solution includes the low-rank constraint solution and the lowest-rank solution.

The idea of low-rank constraint solution comes from the so-called curse of dimensionality. With the explosion of the information in modern society and the growing complexity in practical control problems, the scale of problems is becoming larger and larger and the storage of these large-scale data eventually becomes problematic. A popular and reasonable algebraic technique for alleviating this is to approximate the solution by some low-rank matrix. In this way, there already exists a significant number of low-rank methods for solving Lyapunov equations using this principle. We refer to [12,15,20,21,23,25,35,37,42] for details. Among most of these papers, the generated low-rank matrix solution is achieved as an approximation to the original unique solution when the corresponding Lyapunov equation is nonsingular, and the accuracy of the best low-rank approximation increases rapidly with growing rank [34]. While in some other cases, the continuous or discrete Lyapunov equation may be singular and hence possesses more than one solution in practical problems. In this case, how to get those minimal rank exact solutions remains essential as well.

The lowest-rank (i.e., minimal rank) solution has wide applications in the bilinear matrix inequality problem, static output feedback stabilization, reduced-order H^∞ synthesis, and μ-synthesis with constant scaling, see [11,28,29,38,39]. All these problems can be mathematically formulated as rank minimization problems. Technically, however, rank minimization problems are NP-hard in general due to the non-continuity and non-convexity of the rank function. A common idea is to construct some easy-tackling relaxations. A variety of heuristic algorithms based on local optimization then emerged, such as the alternating projection and its variations [10,31], the alternating matrix inequalities technique [41], linearization [13], augmented Lagrangian methods [6], and the nuclear-norm method [7]. Particularly, when the matrix variable is symmetric and positive semidefinite, the nuclear norm turns out to be the trace function and the corresponding heuristic is the so-called trace norm heuristic. This method has been observed to produce very low-rank solutions in practice and especially it can provide the exact solution for the discrete Lyapunov inequalities with semidefinite cone constraints