Linear Algebra and its Applications 452 (2014) 263–269

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the typical rank of real bivariate polynomials

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ARTICLE INFO

Article history:
Received 14 April 2012
Accepted 1 April 2014
Available online 16 April 2014
Submitted by L.-H. Lim

MSC:
14N05
15A69

Keywords:
Typical rank
Real bivariate polynomial
Symmetric tensor rank
Bivariate homogeneous polynomial

ABSTRACT

Here we study the typical rank for real bivariate homogeneous polynomials of degree $d \geqslant 6$ (the case $d \leqslant 5$ being settled by P. Comon and G. Ottaviani). We prove that $d-1$ is a typical rank and that if d is odd, then $(d+3)/2$ is a typical rank. The Comon–Ottaviani conjecture was later completely solved by G. Blekherman.

1. Introduction

For any integer $d \geqslant 0$ let $\mathbb{K}[x,y]_d$, $\mathbb{K}$ either $\mathbb{C}$ or $\mathbb{R}$, denote the $(d+1)$-dimensional $\mathbb{K}$-vector space of all degree d bivariate homogeneous polynomials. For any $f \in \mathbb{R}[x,y]_d$ (resp. $f \in \mathbb{C}[x,y]_d$) let $Rsr(f)$ (resp. $Csr(f)$) denote the minimal integer r such that $f = \sum_{i=1}^{d} c_i L_i^d$ with $c_i \in \mathbb{R}$ and $L_i \in \mathbb{R}[x,y]_1$ (resp. $c_i \in \mathbb{C}$ and $L_i \in \mathbb{C}[x,y]_1$). The positive integer $Rsr(f)$ (resp. $Csr(f)$) is called the real (resp. complex) rank of f. If $f \in \mathbb{R}[x,y]_d$, then both $Rsr(f)$ and $Csr(f)$ are defined. Obviously $Rsr(f) \geqslant Csr(f)$. Quite often strict inequality holds (see e.g. [7] and references therein). The computation

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[1] The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

http://dx.doi.org/10.1016/j.laa.2014.04.001
0024-3795/

of the integer $Rsr(f)$ is used in real-life applications ([4,8,10,12] and the introductions of [3,11,13]). However, in many cases coming from Engineering the coefficients of f are known only approximatively. Unfortunately, $Rsr(f)$ is neither upper semicontinuous not lower semicontinuous. Over $\mathbb{C}$ it is known the existence of a non-empty Zariski open subset $\mathcal{U}$ of $\mathbb{C}[x,y]_d \setminus \{0\}$ such that $Csr(f) = \lfloor (d+2)/2 \rfloor$ for every $f \in \mathcal{U}$ ([7], [9, §I.3]). We recall that Zariski open implies that $\mathcal{U}$ is dense in $\mathbb{C}[x,y]_d$ for the euclidean topology and that $\mathbb{C}[x,y]_d \setminus \mathcal{U}$ is a union of finitely many differentiable manifolds with real codimension at least 2. In the complex case much more is known, even for $f \in \mathbb{C}[x,y]_d \setminus \mathcal{U}$ ([9, §1.3], [10, 9.2.2], [6], [3, §3], [11, Theorem 4.1]). In the real case the picture is more complicated, because $\mathcal{U} \cap \mathbb{R}[x,y]_d$ may have several connected components. An integer $t > 0$ is called a *typical rank* in degree d if there is a non-empty open subset V (for the euclidean topology) of the real vector space $\mathbb{R}[x,y]_d$ such that $Rsr(f) = t$ for all $f \in V$. The existence of $\mathcal{U} \subset \mathbb{C}[x,y]_d$ such that $Csr(g) = \lfloor (d+2)/2 \rfloor$ for all $g \in \mathcal{U}$ implies that any typical rank is at least $\lfloor (d+2)/2 \rfloor$. If t is a typical rank in degree d, then $t \leqslant d$ [7, Proposition 2.1]. It is well-known that $\lfloor (d+2)/2 \rfloor$ and d are typical ranks and there is a clear description of a euclidean open subset of $\mathbb{R}[x,y]_d$ parametrizing polynomials with real rank d: the set of all polynomials with d distinct real roots ([7, Proposition 3.4], [5, Corollary 1]). P. Comon and G. Ottaviani found all typical ranks for $d \leqslant 5$ and conjectured that all integers t such that $\lfloor (d+2)/2 \rfloor \leqslant t \leqslant d$ are typical ranks for real bivariate forms of degree d.

In this note we prove the following results.

Theorem 1. *For each $d \geqslant 5$ the integer $d-1$ is a typical rank for real bivariate degree d forms.*

Theorem 2. *Fix an odd integer $d = 2m+1 \geqslant 5$. Then $m+1$, $m+2$ and $2m+1$ are typical ranks for real degree d bivariate forms.*

It is well-known that $\lfloor (d+2)/2 \rfloor$ is always a typical rank. This observation, [7] and Theorems 1, 2 prove Comon–Ottaviani conjecture if $d \leqslant 7$, that for any even $d \geqslant 6$ there are at least 3 typical ranks and that for every odd $d \geqslant 7$ there are at least 4 typical ranks.

After the submission of this paper G. Blekherman posted his paper [2] in which he proves Comon–Ottaviani conjecture.

2. The proofs

In this paper "Linear subspaces" are projective subspaces of an ambient projective space (over $\mathbb{C}$ or over $\mathbb{R}$) and "dim" and "$\dim_{\mathbb{R}}$" always denote dimensions of projective spaces.

For any $f \in \mathbb{R}[x,y]_d \setminus \{0\}$ and any $c \in \mathbb{R} \setminus \{0\}$ we have $Rsr(f) = Rsr(cf)$. Hence the question about the real rank is a question concerning polynomials, up to a non-zero scalar