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Interpolation problems for holomorphic functions ☆

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ABSTRACT

In this note, we discuss the Nevanlinna–Pick problem, corona problem and Carathéodory–Fejér problem for bounded holomorphic functions.

1. Introduction

The classical Nevanlinna–Pick problem is to find a holomorphic function from open unit disk to open unit disk taking given points to given points. More precisely, let $\{z_1, \ldots, z_n\}$ and $\{w_1, \ldots, w_n\}$ be two collections of complex numbers in the open unit

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disk $\mathbb{D}$. This is the problem of finding a holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(z_i) = w_i$, for all $i = 1, \dots, n$. This problem was independently solved by G. Pick in 1916 and R. Nevanlinna in 1919, respectively. It was proved that such a function exists if and only if the *Pick matrix*

$$\left(\frac{1 - \bar{w}_i w_j}{1 - \bar{z}_i z_j}\right)_{i,j=1}^n$$

is positive semi-definite. Since then, many types of this problem have been studied over the last few decades. For example, J. Agler extended this problem to the bidisk as follows. Let $\{\alpha_1, \dots, \alpha_n\}$, $\{\beta_1, \dots, \beta_n\}$ and $\{z_1, \dots, z_n\}$ be collections of points in $\mathbb{D}$. There is a holomorphic function $f : \mathbb{D}^2 \to \overline{\mathbb{D}}$ such that $f(\alpha_i, \beta_i) = z_i$ if and only if there are positive semi-definite matrices Γ and Δ in M_n such that

$$(1 - \overline{z_i} z_j) = (1 - \overline{\alpha_i}\alpha_j)\Gamma_{ij} + (1 - \overline{\beta_i}\beta_j)\Delta_{ij}.$$

J. Agler and J.E. McCarthy solved in [3] the matrix-valued Nevanlinna–Pick problem on the bidisk as follows. Let $A_1, \dots, A_n$ be $k \times k$ matrices. They showed that there is a holomorphic function $f : \mathbb{D}^2 \to M_k$ with $\|f\| \leqslant 1$ such that $f(\alpha_i, \beta_i) = A_i$ if and only if there are positive semi-definite matrices Γ and Δ in M_{nk} such that

$$\left(I_k - A_i^* A_j\right)_{l;m} = (1 - \overline{\alpha_i}\alpha_j)\Gamma_{i,l;j,m} + (1 - \overline{\beta_i}\beta_j)\Delta_{i,l;j,m}.$$

An operator-valued holomorphic function F on the polydisk $\mathbb{D}^d$ is said to be in the *Schur–Agler class* if

$$\left\|F(T_1, \dots, T_d)\right\| < 1$$

for any commuting d-tuple $(T_1, \dots, T_d)$ of strict contraction operators on a Hilbert space. J.A. Ball and T.T. Trent studied in [13] the operator-valued Nevanlinna–Pick problems on the polydisk. One of them is as follows. Given n distinct points $\{w^j = (w_1^j, \dots, w_d^j)\}$ in $\mathbb{D}^d$, n operators $\{u_j\}$ in the space $B(N_j, H)$ of bounded linear operators between Hilbert spaces, and n operators $\{v_j\}$ in $B(N_j, K)$. Then there is a holomorphic function $F : \mathbb{D}^d \to B(H, K)$ in the Schur–Agler class such that

$$F(w_j)u_j = v_j, \quad \forall j = 1, \dots, n,$$

if and only if there exist positive semi-definite $n \times n$ matrices M^k with matrix entries M_{ij}^k in $B(N_j, N_i)$ such that

$$u_i^* u_j - v_i^* v_j = \sum_{k=1}^d \left(1 - \overline{w}_k^i w_k^j\right) N_{ij}^k.$$