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Three distance characteristic polynomials of some graphs ☆

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A R T I C L E I N F O

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A B S T R A C T

We give complete information about the distance, distance Laplacian and distance signless Laplacian characteristic polynomials of graphs obtained by a generalized join graph operation on families of graphs. As an application of these results, we construct many pairs of nonisomorphic distance, distance Laplacian and distance signless Laplacian cospectral graphs, and then give a negative answer to the question "Can every connected graph be determined by its distance Laplacian spectrum and/or distance signless Laplacian spectrum?" proposed in Aouchiche and Hansen (2013) [2].

1. Introduction

The distance matrix is more complex than the ordinary adjacency matrix of a graph since the distance matrix is a complete matrix (dense) while the adjacency matrix often is very sparse. Thus the computation of the characteristic polynomial of the distance matrix is computationally a much more intense problem and, in general, there are no

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simple analytical solutions except for a few trees. The distance matrix of a graph has numerous applications to chemistry and other branches of science. The distance matrix, containing information on various walks and self-avoiding walks of chemical graphs, is immensely useful in the computation of topological indices such as the Wiener index, is useful in the computation of thermodynamic properties such as pressure and temperature coefficients and it contains more structural information compared to a simple adjacency matrix. For a survey see [1] and also the papers cited therein.

All graphs considered here are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix whose (i, j)-entry is 1 if i and j are adjacent and 0 otherwise. Let $D(G)$ be the *diagonal degree matrix* of G, where the (i, i)-entry is equal to $d_G(i)$, the degree of vertex i. Then the *Laplacian matrix* of G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$.

For $i, j \in V(G)$, the distance between i and j, denoted by $d_G(i, j)$, is the length of a shortest path from i to j in G. The *distance matrix* of a connected graph G is the $n \times n$ matrix $\mathcal{D}(G) = (d_G(i, j))$. The *transmission* $Tr(i)$ of a vertex i is the sum of the distances from i to all other vertices, i.e., $Tr(i) = \sum_{j=1}^{n} d_G(i, j)$. A connected graph is said to be *distance regular* if the transmission is a constant for every vertex. Let $\mathcal{T}(G)$ be the *diagonal transmission matrix* of G, where the (i, i)-entry is equal to $Tr(i)$. Similarly to the Laplacian and signless Laplacian, Aouchiche and Hansen [2] defined the *distance Laplacian* and *distance signless Laplacian* of a connected graph G as the matrices $\mathcal{D}^L(G) = \mathcal{T}(G) - \mathcal{D}(G)$ and $\mathcal{D}^Q(G) = \mathcal{T}(G) + \mathcal{D}(G)$, respectively.

For a graph G, as we see, there are many matrices associated with G. Let $M = M(G)$ be a matrix associated with G. The M-polynomial is defined as $\Phi_M(G, x) = \det(xI - M)$, where I is the identity matrix. The M-eigenvalues are the roots of the M-polynomial, and the M-spectrum of G is a multiset consisting of the M-eigenvalues. A graph is called M-integral if its M-spectrum consists only of integers. Graphs with the same M-spectrum are called *M-cospectral graphs*. Two M-cospectral non-isomorphic graphs G and H are called *M-cospectral mates* or *M-mates*. Aouchiche and Hansen [2] propose the following question.

Question 1.1. Can every connected graph be determined by its $\mathcal{D}^L$-spectrum and/or $\mathcal{D}^Q$-spectrum?

For two disjoint graphs G and H, let $G \cup H$ denote the union of G and H. And let $G \vee H$ be the graph obtained from $G \cup H$ by joining every vertex of G to every vertex of H. The union and join may be viewed as special cases of a more general operation which are called "generalization composition" in [3]. If G is labeled and has k vertices, then the graph $G[H_1, \ldots, H_k]$ is formed by taking the disjoint graphs $H_1, \ldots, H_k$ and then joining every vertex of H_i to every vertex of H_j when i is adjacent to j in G. Thus the join is given by $G \vee H = K_2[G, H]$. Schwenk [3] and Cardoso et al. [4] provided complete information about the A-spectrum of $G[H_1, \ldots, H_k]$ for any graph G and regular graphs $H_1, \ldots, H_k$.