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On the equality of generalized matrix functions ☆

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ABSTRACT

Let H and K be arbitrary subgroups of the symmetric group S_n and let φ and ψ be irreducible characters of H and K, respectively. The main result of this paper is that the two generalized matrix functions d_φ^H and d_ψ^K are equal on the set of singular matrices if and only if $\varphi_{H\cap K} = \psi_{H\cap K}$ and both of φ and ψ vanish outside of $H \cap K$.

1. Introduction

Let S_n be the symmetric group of degree n, G an arbitrary subgroup of S_n, and $\chi : G \to \mathbb{C}$ a complex valued function defined on G. Also denote by $M_n(\mathbb{C})$ the set of all n-by-n matrices over $\mathbb{C}$. We define the function $d_\chi^G : M_n(\mathbb{C}) \to \mathbb{C}$ as follows:

$$d_\chi^G(A) = \sum_{\sigma\in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $A = (a_{ij}) \in M_n(\mathbb{C})$. The function d_χ^G is called the *generalized matrix function* associated with G and χ. Note that if $G = S_n$ and $\chi = 1_G$ is the principal character of G, then d_χ^G is the permanent and if $G = S_n$ and $\chi = \varepsilon$ is the alternating character of G, then d_χ^G is the determinant. It is trivial that if χ and φ are two complex valued functions defined on G and $\lambda \in \mathbb{C}$, then $d_{\chi+\lambda\varphi}^G = d_\chi^G + \lambda d_\varphi^G$. We refer

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the reader to [2] and [3] for more deep information about generalized matrix functions. Throughout, let $\hat{\chi}$ be an extension of χ to S_n which vanishes outside of G. It is obvious that $d_\chi^G = d_{\hat{\chi}}^{S_n}$.

In this paper we give a necessary and sufficient condition for the equality of two generalized matrix functions. In particular, we show that if $H \subseteq K$ are subgroups of S_n, and if φ and ψ are irreducible characters of H and K, respectively, then $d_\varphi^H(A) = d_\psi^K(A)$ for all singular matrices A if and only if $H = K$ and $\varphi = \psi$.

2. Main results

For $\sigma \in S_n$, let $\mathrm{Fix}(\sigma) = \{i : 1 \leqslant i \leqslant n, (i)\sigma = i\}$ be the set of fixed points of σ and $l(\sigma) = n - |\mathrm{Fix}(\sigma)|$. Note that for all $\sigma \in S_n$, we have $l(\sigma) \neq 1$. Also, let A_σ be the permutation matrix induced by σ and E_{ij} be an standard matrix unit, i.e. the matrix which has 1 in the (i, j)th entry and zeros elsewhere. The following theorem whose proof is based on a beautiful induction shows that the only generalized matrix functions which act like the determinant are the scalar multiples of the determinant.

Theorem 2.1. *Let $G \leqslant S_n$ and $\chi : G \to \mathbb{C}$ be a nonzero function. Then the following are equivalent:*
(i) $d_\chi^G(A) \neq 0$ for all nonsingular matrices A;
(ii) $d_\chi^G(A) = 0$ for all singular matrices A;
(iii) $G = S_n$ and $\chi = \chi(1)\varepsilon$.

Proof. If (iii) holds, then $d_\chi^G = \chi(1)\det$ and so we obtain (i) and (ii).
Now we may assume that the case (i) or (ii) holds. First we claim that $\hat{\chi}(\sigma) = \chi(1)\varepsilon(\sigma)$ for all $\sigma \in S_n$. Note that in case (i), for all $\sigma \in S_n$,

$$0 \neq d_\chi^G(A_\sigma) = d_{\hat{\chi}}^{S_n}(A_\sigma) = \hat{\chi}(\sigma),$$

showing that $G = S_n$, and so $\hat{\chi} = \chi$.
We now prove the claim by induction on $l(\sigma)$. If $l(\sigma) = 0$, then $\sigma = 1$ and the result follows. Suppose that $l(\sigma) \geqslant 2$ and the assertion is true for all $\tau \in S_n$ with $l(\tau) < l(\sigma)$. Let $\sigma = \sigma_1 \cdots \sigma_r$ be the decomposition of σ into the nontrivial disjoint cycles and let $\sigma_1 = (a_1 \cdots a_s)$. By choosing the permutation τ to be $(a_1 a_2)\sigma$ we have that σ is even iff τ is odd, and so $\varepsilon(\sigma) = -\varepsilon(\tau)$. Moreover, $l(\tau) = l(\sigma) - 2$ if $s = 2$, and $l(\tau) = l(\sigma) - 1$ otherwise. Thus, by induction, we have $\hat{\chi}(\tau) = \chi(1)\varepsilon(\tau)$. Define the matrix A as follows:

$$A = \begin{cases} A_\sigma + \alpha E_{a_1 a_1} + E_{a_2 a_2} & \text{if } s = 2 \\ A_\sigma + E_{a_1 a_3} + \alpha E_{a_2 a_2} & \text{otherwise} \end{cases}.$$

Since $\det(A) = \varepsilon(\sigma) + \alpha\varepsilon(\tau)$, so A is nonsingular iff $\alpha \neq 1$. Also,

$$d_\chi^G(A) = d_{\hat{\chi}}^{S_n}(A) = \hat{\chi}(\sigma) + \alpha\hat{\chi}(\tau).$$

In case (i), we have $\chi(\sigma) = -\chi(\tau)$ since otherwise by taking $\alpha = -\chi(\sigma)/\chi(\tau)$, we obtain $d_\chi^G(A) = 0$, which is a contradiction. Therefore,

$$\chi(\sigma) = -\chi(\tau) = -\chi(1)\varepsilon(\tau) = \chi(1)\varepsilon(\sigma),$$

completing the proof of the theorem in this case.
In case (ii) by taking $\alpha = 1$, we have $d_\chi^G(A) = 0$. Therefore,

$$\hat{\chi}(\sigma) = -\hat{\chi}(\tau) = -\chi(1)\varepsilon(\tau) = \chi(1)\varepsilon(\sigma),$$

completing the proof of the claim. Now since χ is a nonzero function, we have $\chi(1) \neq 0$, and so $G = S_n$ and $\chi = \chi(1)\varepsilon$. $\square$