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Jensen and Minkowski inequalities for operator means and anti-norms

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ABSTRACT

Jensen inequalities for positive linear maps of Choi and Hansen–Pedersen type are established for a large class of operator/matrix means such as some p-means and some Kubo–Ando means. These results are also extensions of the Minkowski determinantal inequality. To this end we develop the study of anti-norms, a notion parallel to the symmetric norms in matrix analysis, including functionals like Schatten q-norms for a parameter $q \in (-\infty, 1]$ and the Minkowski functional $\det^{1/n} A$.

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1. Introduction

Jensen inequalities for matrices and operators have various versions. The most general ones involve a unital positive linear map $\mathbb{E} : \mathbb{M}_n \to \mathbb{M}_m$. For instance, if $f(t)$ is operator concave on an interval Ω, then

$$f\big(\mathbb{E}(Z)\big) \geq \mathbb{E}\big(f(Z)\big) \tag{1.1}$$

for all $Z \in \mathbb{M}_n\{\Omega\}$, the $n \times n$ Hermitian matrices with spectra in Ω. This is Choi's inequality [12], which is specialized to Hansen–Pedersen's inequality [13]

$$f\left(\sum_{i=1}^{k} C_i^* Z_i C_i\right) \geq \sum_{i=1}^{k} C_i^* f(Z_i) C_i \tag{1.2}$$

for C^*-convex combinations in $\mathbb{M}_n\{\Omega\}$ with $n \times m$ matrices C_i such that $\sum_{i=1}^{k} C_i^* C_i = I$, the identity. These Jensen's inequalities are famous characterizations of operator concavity of the function f:

$$f\left(\frac{A+B}{2}\right) \geq \frac{f(A)+f(B)}{2}, \quad A, B \in \mathbb{M}_n\{\Omega\}. \tag{1.3}$$

Are there similar inequalities by making use of the pth power map $\mathbb{E}_p(Z) := \mathbb{E}^{1/p}(Z^p)$ with $p > 0$? We deal with this question in Section 2. This contains some Jensen type inequalities for the *power p-means*

$$A\,\beta_p\,B := \left(\frac{A^p + B^p}{2}\right)^{1/p} \tag{1.4}$$

of two positive operators A, B.

An axiomatic theory on *operator means* (for positive operators on a Hilbert space) was developed by Kubo and Ando [18] related to operator monotone functions. (See [4] for operator monotone and operator convex functions.) Each operator mean σ is associated with a non-negative operator monotone function $h(t)$ on $[0, \infty)$ with $h(1) = 1$, called the *representing function* of σ, in such a way that

$$A\,\sigma\,B = A^{1/2} h\big(A^{-1/2} B A^{-1/2}\big) A^{1/2} \tag{1.5}$$

for every invertible $A, B \in \mathbb{M}_n^+$, the $n \times n$ positive semi-definite matrices. The above expression is further extended to general $A, B \in \mathbb{M}_n^+$ as

$$A\,\sigma\,B = \lim_{\varepsilon \searrow 0} (A + \varepsilon I)\,\sigma\,(B + \varepsilon I). \tag{1.6}$$