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Matrix roots of eventually positive matrices

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ABSTRACT

Eventually positive matrices are real matrices whose powers become and remain strictly positive. As such, eventually positive matrices are *a fortiori* matrix roots of positive matrices, which motivates us to study the matrix roots of primitive matrices. Using classical matrix function theory and Perron–Frobenius theory, we characterize, classify, and describe in terms of the real Jordan canonical form the pth-roots of eventually positive matrices.

1. Introduction

A matrix $A \in M_n(\mathbb{R})$ is *eventually positive* (*nonnegative*) if there exists a nonnegative integer p such that A^k is entrywise positive (nonnegative) for all $k \geqslant p$. If p is the smallest such integer, then p is called the *power index of* A and is denoted by $p(A)$.

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Eventually nonnegative matrices have been the subject of study in several papers [1–8] and it is well known that the notions of eventual positivity and nonnegativity are associated with properties of the eigenspace corresponding to the spectral radius.

A matrix $A \in M_n(\mathbb{R})$ has the *Perron–Frobenius property* if its spectral radius is a positive eigenvalue corresponding to an entrywise nonnegative eigenvector. The *strong Perron–Frobenius property* further requires that the spectral radius is simple; that it dominates in modulus every other eigenvalue of A; and that it has an entrywise positive eigenvector.

Several challenges regarding the theory and applications of eventually nonnegative matrices remain unresolved. For example, eventual positivity of A is equivalent to A and A^T having the strong Perron–Frobenius property, however, the Perron–Frobenius property for A and A^T is a necessary but not sufficient condition for eventual nonnegativity of A.

An eventually nonnegative (positive) matrix with power index $p = p(A)$ is, *a fortiori*, a pth-root of the nonnegative (positive) matrix A^p. As a consequence, in order to gain more insight into the powers of an eventually nonnegative (positive) matrix, it is only natural to examine the roots of matrices that possess the (strong) Perron–Frobenius property. We begin this pursuit herein by characterizing the roots of matrices that possess the strong Perron–Frobenius property.

We proceed as follows: in Section 2, we recall results concerning matrix functions and, for the sake of completeness and clarity, we present facts needed to analyze a matrix function via the real Jordan canonical form; we also use the real Jordan canonical form to give alternate proofs for [9, Theorems 2.3 and 2.4]. In Section 3, we recall results from the Perron–Frobenius theory of nonnegative matrices and (eventually) positive matrices. We characterize the eventually positive roots of a general primitive matrix, and illustrate our main results via examples. We also present a related result concerning *eventually stochastic* matrices.

2. Matrix roots via the complex and real Jordan canonical form

We review some basic notions and results from the theory of matrix functions (for further results, see [10], [11, Chapter 9], or [12, Chapter 6]).

Let $J_n(\lambda) \in M_n(\mathbb{C})$ denote the $n \times n$ Jordan block with eigenvalue λ. For $A \in M_n(\mathbb{C})$, let $J = Z^{-1}AZ = \bigoplus_{i=1}^{t} J_{n_i}(\lambda_i) = \bigoplus_{i=1}^{t} J_{n_i}$, where $\sum n_i = n$, denote its Jordan canonical form. Denote by $\lambda_1, \dots, \lambda_s$ the *distinct* eigenvalues of A, and, for $i = 1, \dots, s$, let m_i denote the *index* of λ_i, i.e., the size of the largest Jordan block associated with λ_i. Denote by i the imaginary unit, i.e., $\mathrm{i} := \sqrt{-1}$.

Definition 2.1. Let $f : \mathbb{C} \to \mathbb{C}$ be a function and let $f^{(k)}$ denote the kth derivative of f. The function f is said to be *defined on the spectrum of A* if the values

$$f^{(k)}(\lambda_i), \quad k = 0, \dots, m_i - 1, \; i = 1, \dots, s,$$

called *the values of the function f on the spectrum of A*, exist.

Definition 2.2 *(Matrix function via Jordan canonical form).* If f is defined on the spectrum of $A \in M_n(\mathbb{C})$, then

$$f(A) := Zf(J)Z^{-1} = Z\left(\bigoplus_{i=1}^{t} f(J_{n_i})\right)Z^{-1},$$

where

$$f(J_{n_i}) := \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \dots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\ & f(\lambda_i) & \ddots & \vdots \\ & & \ddots & f'(\lambda_i) \\ & & & f(\lambda_i) \end{bmatrix}. \tag{2.1}$$