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A straightforward proof of the polynomial factorization of a positive semi-definite polynomial matrix

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ABSTRACT

In this paper we present a straightforward proof of the well-known fact that a positive semi-definite polynomial matrix can be written as the product of a polynomial matrix and its associated (Hermitian) transposed. For polynomial matrices with complex coefficients both factors also have complex coefficients and can be taken of the same size as the original matrix. For polynomial matrices with real coefficients both factors may have real coefficients, but then the left factor may need twice as many columns as the original matrix, and analogously for the right factor.

1. Introduction and notation

The goal of this paper is to give a straightforward proof of the well-known fact that a positive semi-definite polynomial matrix can be written as the product of a polynomial matrix and its associated (Hermitian) transposed, see for instance [3] or [5]. We do this for polynomial matrices with complex coefficients first, see Theorem 1. From the obtained factorization, we derive a factorization for polynomial matrices with real coefficients only, see Corollary 2.

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Our proof consists of basic linear algebra and is inspired by [4], where also a simple proof is provided, however there the proof is based on complex function theory.

Before stating Theorem 1 and Corollary 2 with their proofs, we present some notation and recall some basic facts from linear algebra.

In this paper we write $\mathbb{R}^{k\times l}$ and $\mathbb{C}^{k\times l}$ for the set of $k \times l$ matrices with real and complex coefficients, respectively. The sets of $k \times l$ polynomial matrices with real and complex coefficients are denoted by $\mathbb{R}^{k\times l}[x]$ and $\mathbb{C}^{k\times l}[x]$, respectively. Throughout this paper the indeterminate x will mainly be real, although we occasionally also will evaluate a polynomial matrix at a complex value for x.

For a matrix $M \in \mathbb{R}^{k\times l}$, we denote the transposed by $M^\top \in \mathbb{R}^{l\times k}$ and for a matrix $M \in \mathbb{C}^{k\times l}$, we write $M^* \in \mathbb{C}^{l\times k}$ for its conjugate transposed, i.e., $M^* = \overline{M}^\top$, where $\overline{M}$ denotes the entry-wise complex conjugate of M.

Given a matrix $M \in \mathbb{C}^{k\times l}$, we write $\mathcal{N}(M)$ for its nullspace in $\mathbb{C}^l$ and $\mathcal{R}(M)$ for its column space in $\mathbb{C}^k$. We use the standard inner product in $\mathbb{C}^l$, given by $\langle v, w\rangle := v^*w$ for all $v, w \in \mathbb{C}^l$. Then, writing $^\perp$ for the associated orthogonal complement, it follows that $\mathcal{R}(M^*) = (\mathcal{N}(M))^\perp$, see for instance [2].

Consider a polynomial matrix $P(x) \in \mathbb{C}^{k\times l}[x]$, written as $P(x) = \sum_{j=0}^m P_j x^j$, with coefficient matrices $P_j \in \mathbb{C}^{k\times l}$ for $j = 0, 1, \dots, m$. Then the 'complex transposed' of $P(x)$, denoted $P^*(x)$, is defined as $P^*(x) := \sum_{j=0}^m P_j^* x^j \in \mathbb{C}^{l\times k}[x]$. Observe that for a given value $\gamma \in \mathbb{C}$, there holds $P^*(\gamma) = (P(\bar{\gamma}))^*$. If $P(x) \in \mathbb{R}^{k\times l}[x]$, then $P^*(x)$ will also be denoted as $P^\top(x)$, expressing that $P(x)$ only has real coefficient matrices.

Note that if $P(x) = P^*(x)$, then $P(x)$ is square and the coefficients of $P(x)$ are Hermitian matrices. If in addition $k = 1$, the coefficients of $P(x)$ are scalars, that in fact are real. In case $P(x) \in \mathbb{R}^{k\times k}[x]$ with $P(x) = P^*(x)$, the coefficients of $P(x)$ are symmetric matrices.

Given a polynomial matrix $P(x) \in \mathbb{C}^{k\times k}[x]$ such that $P(x) = P^*(x)$, we say that $P(x)$ is positive semi-definite, denoted as $P(x) \succcurlyeq 0$, when $v^*P(x)v \geqslant 0$ for all $x \in \mathbb{R}$ and $v \in \mathbb{C}^k$. This definition also is valid in case $P(x)$ has real coefficient matrices only, or in case $P(x)$ actually is a constant matrix $M \in \mathbb{C}^{k\times k}$. In this paper, whenever we write that $P(x) \succcurlyeq 0$ for a given $P(x) \in \mathbb{C}^{k\times k}[x]$, we assume that $P(x) = P^*(x)$. Note that this interpretation of $P(x) \succcurlyeq 0$ also makes sense in case $P(x) \in \mathbb{R}^{k\times k}[x]$.

In the next section, we shall make use of the Schur complement. To explain this in some more detail, consider a polynomial matrix $P(x) \in \mathbb{C}^{k\times k}$ with $k > 1$. Assume further that $P(x) \succcurlyeq 0$ and that $P(x)$ is written as

$$P(x) = \begin{bmatrix} a(x) & g^*(x) \\ g(x) & C(x) \end{bmatrix},$$

with $a(x)$ a scalar polynomial, $C(x) \in \mathbb{C}^{(n-1)\times(n-1)}[x]$ and $g(x) \in \mathbb{C}^{(n-1)\times 1}[x]$. Because $P(x) = P^*(x)$, it follows that $C(x) = C^*(x)$ and $a(x) = a^*(x)$, the latter implying that the coefficients of $a(x)$ are real. Since $P(x) \succcurlyeq 0$, also $C(x) \succcurlyeq 0$ and $a(x) \succcurlyeq 0$. From the latter it follows that $a(x) \geqslant 0$ for all $x \in \mathbb{R}$.