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Weighted inductive means

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ABSTRACT

In this paper we present a unified framework for weighted inductive means on the cone $\mathbb{P}$ of positive definite Hermitian matrices as natural multivariable extensions of two variable weighted means, particularly of metric midpoint operations on $\mathbb{P}$. It includes some well-known multivariable weighted matrix means: the weighted arithmetic, harmonic, resolvent, Sturm's inductive geometric mean on the Riemannian manifold $\mathbb{P}$ equipped with the trace metric, Log-Euclidean and spectral geometric means. A recursion (or weight additive) formula is derived and applied to find a closed form and basic properties for a weighted inductive mean. An upper bound on the sensitivity, a metric characterization and min and max optimization problems over permutations for the inductive geometric mean are presented. Moreover, we apply the obtained results to a class of midpoint operations of the non-positively curved Hadamard metrics on $\mathbb{P}$ parameterized over Hermitian unitary matrices.

1. Introduction

The convex combination $A(\omega; a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i$ on a vector space, where $\omega = (w_1, \ldots, w_n)$ is a positive probability vector, plays an important role in the study of

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convex analysis and convex geometry. It is also called the ω-weighted arithmetic mean of $a_1, \ldots, a_n$ and can be constructed recursively as follows:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= \frac{w_1}{w_1+w_2}a_1 + \frac{w_2}{w_1+w_2}a_2 = \omega^{(1)}s_1 + \big(1-\omega^{(1)}\big)a_2, \\ &\vdots \\ s_k &= \frac{1}{\sum_{i=1}^k w_i}\sum_{i=1}^k w_i a_i = \omega^{(k-1)}s_{k-1} + \big(1-\omega^{(k-1)}\big)a_k, \end{aligned}$$

and $s_n = \sum_{i=1}^n w_i a_i = \omega^{(n-1)}s_{n-1} + (1-\omega^{(n-1)})a_n$, where

$$\omega^{(k)} = \frac{w_1 + \cdots + w_k}{w_1 + \cdots + w_{k+1}}.$$

Alternatively, the weighted arithmetic mean can be inductively defined by

$$A_1(1;a) = a, \qquad A_n(\omega; a_1, \ldots, a_n) = (1-w_n)A_{n-1}(\omega_{n-1}; a_1, \ldots, a_{n-1}) + w_n a_n,$$

where $\omega_{n-1} := \frac{1}{1-w_n}(w_1, \ldots, w_{n-1})$. We note that these construction schemes depend only on the two-variable weighted arithmetic mean $A(1-t, t; a, b) = (1-t)a + tb$, the weighted version of the arithmetic mean operation $\frac{1}{2}(a+b)$.

A classical and important problem in the study of matrix means of positive definite matrices is extending two variable (weighted) matrix means to higher order preserving their properties. The weighted arithmetic $\sum_{i=1}^n w_i A_i$ and harmonic $(\sum_{i=1}^n w_i A_i^{-1})^{-1}$ means of n positive definite matrices $A_1, \ldots, A_n$ are basic examples of higher order weighted matrix means. The arithmetic–geometric–harmonic inequalities of two positive definite matrices

$$\big[(1-t)A^{-1} + tB^{-1}\big]^{-1} \leq A \#_t B \leq (1-t)A + tB \quad \big(t \in [0,1]\big)$$

where $A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is the weighted geometric mean of positive definite matrices A and B, suggest a nice problem of constructing multivariable weighted geometric means satisfying a multivariable weighted arithmetic–geometric–harmonic inequalities. Three recent approaches for unweighted case for extending two-variable geometric mean of positive definite matrices to higher order have been developed by Bhatia and Holbrook [5] via "least squares methods", Ando, Li and Mathias [2] and Bini, Meini and Poloni [7] via "symmetrization methods" and induction. Although these geometric means of n-positive definite matrices satisfy almost all properties of the two variable geometric mean including the AGH mean inequalities, it is impossible to find their closed forms.

The main purpose of this paper is to propose a unified method for extending a two variable weighted mean of positive definite matrices to n positive definite matrices with