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From finite line graphs to infinite derived signed graphs

G.R. Vijayakumar

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India

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ABSTRACT

Let X be a subset of $\{\pm\alpha \pm \beta\colon\ \alpha, \beta \in \mathcal{B}$ and $\alpha \neq \beta\}$ where $\mathcal{B}$ is an orthonormal set in an inner product space over $\mathbb{R}$, such that $x \in X \Rightarrow -x \notin X$. Then the signed graph which is defined as described below is called a *derived signed graph*: its vertex set is X; two vertices x, y are joined by a positive (negative) edge when $\langle x, y\rangle$ is positive (negative); when $\langle x, y\rangle = 0$, x, y are not joined. Let $\mathfrak{D}$ denote the family of all derived signed graphs—the order of a member of $\mathfrak{D}$ may be infinite. (The family of all generalized line graphs—line graphs belong to this family—is a subfamily of $\mathfrak{D}$.) Let $\mathfrak{M}$ be the class of all minimal nonderivable signed graphs. [$\mathfrak{M}$ includes the 31 (finite) minimal nongeneralized line graphs computed by various methods in the literature.] In this article, we characterize $\mathfrak{D}$, determine $\mathfrak{M}$ and classify the family of all signed graphs S for which, the following holds: for each finite subset X of $V(S)$, the least eigenvalue of $S[X]$ is at least -2. The third result substantially generalizes the well known result (Cameron et al. (1976) [1]) on classifying the family of all finite (signed) graphs with least eigenvalues $\geqslant -2$.

E-mail address: vijay@math.tifr.res.in.

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1. Preliminaries

For unexplained graph theoretic terms and notation, the reader is referred to [18]. For information on inner product spaces, we rely on [7]. Each graph G considered in this article is simple; its order, denoted by $|G|$, need not be finite. A *signed graph* is a graph in which each edge is termed as positive or negative; a graph from which a signed graph S has been formed, is called the *underlying graph* of S and denoted by S_u. For basic information and various results on the notion of (finite) signed graph, we refer the reader to [19] where this notion is defined in a little more generality: the underlying graph of a signed graph need not be simple. Let x, y be two vertices of a signed graph S; to denote that x and y are joined by a positive edge, we write $\langle\langle x, y\rangle\rangle = 1$; by writing '$\langle\langle x, y\rangle\rangle = -1$' we mean that they are joined by a negative edge whereas '$\langle\langle x, y\rangle\rangle = 0$' implies that they are not adjacent. The signed graph obtained from S by changing the sign of each edge is denoted by $-S$. A signed graph T is called an *innergraph* of S, if T_u is an induced subgraph of S_u and each edge of T_u has the same sign in both signed graphs. To denote that a signed graph S' is an innergraph of S, we write $S' \preccurlyeq S$. If $U \subseteq V(S)$, then the innergraph of S with vertex set U is denoted by $S[U]$. If $X = \{x_1, x_2, \ldots, x_n\}$ is a subset of $V(S)$, then the innergraph T with vertex set X is denoted by $S[x_1, x_2, \ldots, x_n]$ also and the product of the values assigned to all edges of T by the function $\langle\langle *, *\rangle\rangle$ is denoted by $\sigma(T)$ or $\sigma(x_1 x_2 \cdots x_n)$. A signed graph T is called a *triangle* (*quadrangle*) if T_u is a cycle of order three (four). Let T be a finite innergraph of S; if there is a vertex p in $V(S) \setminus V(T)$ such that $|N(p) \cap V(T)|$ is odd, then T is called *odd* in S; otherwise it is *even* in S. We regard any graph G as a signed graph having only positive edges; i.e., it is 'identified' with the signed graph which has no negative edges and whose underlying graph is G.

Normally any term related to graphs is used without any change in its meaning for signed graphs also; however, some terms have to be redefined for signed graphs due to edge-signing; here is one such deviation: The *adjacency matrix* of a finite signed graph S is obtained from that of its underlying graph by replacing each entry which corresponds to a negative edge by -1; i.e., for each $(x, y) \in V(S) \times V(S)$, the entry corresponding to it is $\langle\langle x, y\rangle\rangle$. By an eigenvalue of a finite signed graph, we mean an eigenvalue of its adjacency matrix; since this matrix is real and symmetric, its eigenvalues are real. If S' is a finite signed graph and S'' is an innergraph of S', it can be shown that the least eigenvalue of S' is not more than that of S''. (For a proof, see [6, Theorem 9.1.1].) Let S be a signed graph of any order; the infimum of the least eigenvalues of all finite innergraphs of S—note that by the preceding fact, when $|S| < \infty$, this infimum coincides with the least eigenvalue of S—is defined to be the *least eigenvalue* of S and denoted by $\lambda(S)$. (For graphs, the notion just defined has been termed in [13] as 'least limiting eigenvalue'.)

A notable property of a finite line graph G is that $\lambda(G) \geqslant -2$. (For a proof, see [6, Lemma 8.6.2]. It is easy to verify that infinite line graphs also have this property.) This fact has prompted many authors to study in detail the family of all finite graphs with least eigenvalues $\geqslant -2$. Hoffman has found an important subfamily of this, whose members are