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Schur product techniques for the subnormality of commuting 2-variable weighted shifts ☆,☆☆

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ABSTRACT

In this paper, we study the subnormality of 2-variable weighted shifts using the Schur product techniques in matrices and the convolution of two continuous functions. As a consequence, we find the Berger measure of the subnormal weighted shift obtained from the Schur product of two subnormal weighted shifts. As applications, we first give non-trivial, large classes satisfying the Curto–Muhly–Xia conjecture (see the conjecture given below) for 2-variable weighted shifts. We next show when the 2-hyponormality of 2-variable weighted shifts becomes subnormality.

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1. Introduction

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, where $(A \circ B)_{i,j} := (A)_{i,j}(B)_{i,j}$ for $1 \leq i,j \leq n$. For bounded sequences of positive real numbers $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ and $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$, the Schur product of α and β is defined by $\alpha \circ \beta := \{\alpha_n \beta_n\}_{n=0}^{\infty}$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ [30]. Thus, for given two 1-variable subnormal weighted shifts W_α and W_β, their Schur product $W_\alpha \circ W_\beta$, which we denote by $W_{\alpha\circ\beta}$, is subnormal. That is, for each $k \geq 1$, if W_α and W_β are k-hyponormal 1-variable weighted shifts, then the Schur product $W_{\alpha\circ\beta} \equiv W_\alpha \circ W_\beta$ is also a k-hyponormal 1-variable weighted shift [16]. In an entirely similar way we can define the Schur product of two 2-variable weighted shifts $\mathbf{R} := (R_1, R_2)$ and $\mathbf{S} := (S_1, S_2)$ with, respectively, weights $\boldsymbol{\alpha} \equiv (\alpha^{(1)}_{(k_1,k_2)}, \beta^{(1)}_{(k_1,k_2)})$ and $\boldsymbol{\beta} \equiv (\alpha^{(2)}_{(k_1,k_2)}, \beta^{(2)}_{(k_1,k_2)})$ for $(k_1, k_2) \in \mathbb{Z}^2_+$. That is, we define the Schur product of $\mathbf{R}$ and $\mathbf{S}$ by $\mathbf{R}\circ\mathbf{S} := (R_1 \circ S_1, R_2 \circ S_2)$ with weights $\boldsymbol{\alpha}\circ\boldsymbol{\beta} := (\alpha^{(1)}_{(k_1,k_2)}\alpha^{(2)}_{(k_1,k_2)}, \beta^{(1)}_{(k_1,k_2)}\beta^{(2)}_{(k_1,k_2)})_{k_1,k_2=0}^{\infty}$ (see the weight diagram given in Fig. 1). In [33], the second author of this paper extended the result for 1-variable case given above to 2-variable weighted shifts (see Theorem 3.2 given below).

We recall a well known characterization of subnormality for 2-variable weighted shifts $\mathbf{T} \equiv (T_1, T_2) \equiv W_{(\alpha,\beta)}$ [24], due to C. Berger (cf. [5, III.8.16]): $W_{(\alpha,\beta)}$ admits a commuting normal extension if and only if there is a probability measure μ (called the *Berger measure* of $W_{(\alpha,\beta)}$) defined on the 2-dimensional rectangle $R = [0, a_1] \times [0, a_2]$ (where $a_i := \|T_i\|^2$) such that

$$\gamma_{(k_1,k_2)}(W_{(\alpha,\beta)}) = \int_R s^{k_1} t^{k_2}\, d\mu(s,t), \quad \text{for all } (k_1,k_2) \in \mathbb{Z}^2_+ \quad (\text{called } \textit{Berger theorem}), \tag{1}$$

where $\gamma_{(k_1,k_2)}(W_{(\alpha,\beta)})$ is the moment of order (k_1, k_2) for $W_{(\alpha,\beta)}$ (see (4) given below). If $\mathbf{T} \equiv W_\alpha$, that is, for 1-variable weighted shifts, W_α is subnormal if and only if there exists a probability measure ξ_α supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_{k_1}(W_\alpha) := \alpha_0^2 \cdots \alpha_{k_1-1}^2 = \int s^{k_1} d\xi_\alpha(s)$ for all $k_1 \geq 1$.

By the results in [16, Corollary 2.4], [33, Theorem 2.1] and Berger theorem, it is natural to consider the following problems:

Problem 1.1. (i) If W_α and W_β are both subnormal, then their Schur product $W_{\alpha\circ\beta} \equiv W_\alpha \circ W_\beta$ is subnormal. In this case, what is the Berger measure of $W_{\alpha\circ\beta}$?

(ii) If $\mathbf{R}$ and $\mathbf{S}$ are both subnormal, then their Schur product $\mathbf{R}\circ\mathbf{S}$ is also subnormal. In this case, what is the Berger measure of $\mathbf{R}\circ\mathbf{S}$?

This paper considers the problems given above in Theorems 4.1, 4.2 in Section 4.

We consider an old problem in operator theory, the so-called Lifting Problem for Commuting Subnormals (LPCS): given a commuting pair of subnormal operators on a