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Self-adjoint extensions for discrete linear Hamiltonian systems

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ABSTRACT

This paper is concerned with self-adjoint extensions for a class of discrete linear Hamiltonian systems. By applying the generalized von Neumann theory and the GKN theory for Hermitian subspaces, a complete characterization of all the self-adjoint subspace extensions for the systems are obtained in terms of boundary conditions via linear independent square summable solutions. As a consequence, characterizations of all the self-adjoint subspace extensions are given in the limit point and limit circle cases. In addition, some sufficient conditions for the corresponding minimal subspace to be an operator or a densely defined operator are given, and consequently, characterizations of all the self-adjoint operator extensions for the system are obtained. In particular, even-order formally self-adjoint vector difference equations are discussed as a special case.

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1. Introduction

Consider the following discrete linear Hamiltonian system:

$$J\Delta y(t) = \big(P(t) + \lambda W(t)\big)R(y)(t), \quad t \in \mathcal{I}, \tag{1.1}$$

where $\mathcal{I} := \{t\}_{t=a}^{b}$ is an integer interval, a is a finite integer or $a = -\infty$, and b is a finite integer or $b = +\infty$, $b - a \geq 1$; J is the canonical symplectic matrix, i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ unit matrix; Δ is the forward difference operator, i.e., $\Delta y(t) = y(t+1) - y(t)$; $W(t)$ and $P(t)$ are $2n \times 2n$ Hermitian matrices, and the weight function $W(t) \geq 0$ for $t \in \mathcal{I}$; the partial right shift operator $R(y)(t) = (u^T(t+1), v^T(t))^T$ with $y(t) = (u^T(t), v^T(t))^T$ and $u(t)$, $v(t) \in \mathbb{C}^n$; and λ is a complex spectral parameter.

Throughout the whole paper, we assume that $W(t)$ is of the block diagonal form,

$$W(t) = \operatorname{diag}\big\{W_1(t), W_2(t)\big\},$$

where $W_j(t) \geq 0$ is an $n \times n$ Hermitian matrix, $j = 1, 2$. Let $P(t)$ be blocked as

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix},$$

where $A(t)$, $B(t)$ and $C(t)$ are $n \times n$ matrices, $B(t)$ and $C(t)$ are Hermitian matrices, and $A^*(t)$ is the complex conjugate transpose of $A(t)$. Then (1.1) can be written as

$$\begin{aligned} \Delta u(t) &= A(t)u(t+1) + \big(B(t) + \lambda W_2(t)\big)v(t), \\ \Delta v(t) &= \big(C(t) - \lambda W_1(t)\big)u(t+1) - A^*(t)v(t), \quad t \in \mathcal{I}. \end{aligned}$$

To ensure the existence and uniqueness of the solution of any initial value problem for (1.1), we always assume that

(A_1) $I_n - A(t)$ is invertible in $\mathcal{I}$.

Eq. (1.1) contains the following formally self-adjoint vector difference equation of order $2m$:

$$\sum_{j=0}^{m} (-1)^j \Delta^j \big[p_j(t)\Delta^j z(t-j)\big] = \lambda w(t)z(t), \quad t \in \mathcal{I}, \tag{1.2}$$

where $w(t)$ and $p_j(t)$, $0 \leq j \leq m$, are $l \times l$ Hermitian matrices, $w(t) \geq 0$, and $p_m(t)$ is invertible in $\mathcal{I}$. In fact, by letting $y = (u^T, v^T)^T$ with $u = (u_1^T, u_2^T, \ldots, u_m^T)^T$, $v = (v_1^T, v_2^T, \ldots, v_m^T)^T$, and