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Generalized principal pivot transform

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ABSTRACT

The generalized principal pivot transform is a generalization of the principal pivot transform to the singular case, using the Moore–Penrose inverse. In this article we study some properties of the generalized principal pivot transform. We prove that the Moore–Penrose inverse of a range-symmetric, almost skew-symmetric matrix is almost skew-symmetric. It is shown that the generalized principal pivot transform preserves the rank of the symmetric part of a matrix under some conditions.

1. Introduction

Let A be an $n \times n$ complex matrix partitioned into blocks as $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is invertible. The principal pivot transform of A, with respect to A_{11}, is defined as $ppt(A, A_{11}) = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & (A/A_{11}) \end{pmatrix}$, where $(A/A_{11}) = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the Schur complement of A_{11} in A. The principal pivot transform has an interesting history, which is dealt with in detail in [7].

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In this article we study the notion of principal pivot transform for singular matrices (that is to say, the case when A_{11} is singular). In Section 2, we introduce notation and state some preliminary results. In Section 3, we define the generalized principal pivot transform and discuss its properties. In Section 4, first we prove that the Moore–Penrose inverse of a range-symmetric, almost skew-symmetric matrix is almost skew-symmetric. Then we prove that the generalized principal pivot transform preserves the rank of the symmetric part of the matrix under some conditions. As a particular case we get that the principal pivot transform of an almost skew-symmetric matrix is almost skew-symmetric. Our work generalizes some results from [5,7,8].

2. Notation, definitions and preliminary results

Let $\mathbb{C}^{m\times n}(\mathbb{R}^{m\times n})$ denote the set of all $m\times n$ matrices over the complex (real) numbers. For $A\in\mathbb{C}^{m\times n}$, we denote the adjoint of A, the transpose of A, the range space of A and null space of A by A^*, A^t, $R(A)$ and $N(A)$, respectively.

For a given $A\in\mathbb{C}^{m\times n}$, the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying $AXA=A$, $XAX=X$, $(AX)^*=AX$ and $(XA)^*=XA$ is called the Moore–Penrose inverse of A and is denoted by $A^\dagger$. For a given matrix $A\in\mathbb{C}^{n\times n}$, the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying $AXA=A$, $XAX=X$, and $AX=XA$ is called the group inverse of A and is denoted by $A^\#$. If A is nonsingular, then $A^\#=A^{-1}=A^\dagger$. Unlike the Moore–Penrose inverse, which always exists, the group inverse need not exist for all square matrices. A well known necessary and sufficient condition for the existence of $A^\#$ is that $rank(A)=rank(A^2)$. For complementary subspaces L and M of $\mathbb{C}^n$, the projection (not necessarily orthogonal) of $\mathbb{C}^n$ on L along M will be denoted by $P_{L,M}$. If, in addition, L and M are orthogonal then we denote this by P_L. Some of the well known properties of $A^\dagger$ and $A^\#$ which will be frequently used, are [1]: $R(A^*)=R(A^\dagger)$; $N(A^*)=N(A^\dagger)$; $AA^\dagger=P_{R(A)}$; $A^\dagger A=P_{R(A^*)}$; $R(A)=R(A^\#)$; $N(A)=N(A^\#)$; $AA^\#=P_{R(A),N(A)}$. In particular, if $x\in R(A^*)$ then $x=A^\dagger Ax$ and if $x\in R(A)$ then $x=A^\#Ax$.

Definition 2.1. A matrix $A\in\mathbb{C}^{n\times n}(\mathbb{R}^{n\times n})$ is said to be range-Hermitian (range-symmetric) if $R(A)=R(A^*)(R(A)=R(A^t))$.

The following result is known for range-Hermitian matrices.

Theorem 2.1. *(See [1].) Let $A\in\mathbb{C}^{n\times n}$. Then the following are equivalent:*

(a) A is range-Hermitian,
(b) $N(A)=N(A^*)$,
(c) $A^\dagger=A^\#$.