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Equal entries in totally positive matrices ☆

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ABSTRACT

We show that the maximal number of equal entries in a totally positive (resp. totally nonsingular) n-by-n matrix is $\Theta(n^{4/3})$ (resp. $\Theta(n^{3/2})$). Relationships with point-line incidences in the plane, Bruhat order of permutations, and *TP* completability are also presented. We also examine the number and positionings of equal 2-by-2 minors in a 2-by-n *TP* matrix, and give a relationship between the location of equal 2-by-2 minors and outerplanar graphs.

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1. Introduction

An m-by-n matrix is called totally positive (non-negative), TP (TN), if every minor of it is positive (non-negative). Such matrices play an important role in many facets of mathematics [4,7]. The set of the TN matrices is the closure of the set of the TP matrices, that is, there exists an arbitrarily small perturbation of any given TN matrix that is TP. However, proofs of this typically perturb all, or most, entries, which is usually not necessary. Here, we advance the question of which are the minimal sets of entries that must be perturbed in a given TN matrix that it become TP. Our interest stems, in part, from the likely value of this information in understanding the relationship between TN completable and TP completable patterns. Author Johnson conjectures that the TN completable patterns are properly contained in the TP completable ones. However, the question of "small" perturbing sets is likely of interest for other reasons, such as understanding to topological properties of TN matrices. We are somewhat lax, at times, in the notion of perturbation that we use, however, in that we do not always require our perturbations to be small. For this we use the word "change" in place of "perturb".

Here, we begin the study of minimally perturbing collections by asking which sets of entries in the J matrix (all 1's) need be perturbed. Because of positive diagonal scaling, J may as well be any positive rank 1 matrix. A matrix is called TP_k (TN_k) if every k-by-k submatrix of it is TP (TN). A positive rank 1 matrix is TN and TP_1. A virtue of considering J is due to a very important fact: for any TP_2 matrix, some Hadamard power is TP [4]. So, if J can be perturbed to be TP_2, it can, using the same entries, be changed to TP. Hadamard powering preserves the 1's.

We call a matrix *totally nonsingular*, TNS, if *all* its minors are non-zero. To perturb a matrix with all non-zero entries to a TNS matrix, at least one entry of every 2-by-2 submatrix must be available for perturbation. For this, it is equivalent to consider $(0,1)$-matrices in which no 2-by-2 submatrix of 1's occurs (think of the 1's as unperturbed entries and the 0's as entries that may be perturbed). A certain amount of information about how many 1's such a matrix may have is available [5,8,9,11], and it is known, asymptotically, that this is $O(n^{3/2})$ in the n-by-n case [5]. We show that for a matrix with non-zero entries, perturbation of the entries corresponding to the 0's in any $(0,1)$-matrix with no 2-by-2 matrix of 1's, is sufficient to achieve a TNS matrix. This has the consequence that the maximum number of equal entries in an n-by-n TNS matrix is $\Theta(n^{3/2})$.

We say that a $(0,1)$-matrix A is (TP-)*changeable* if there is a change of the entries of J, corresponding to the 0's in A, such that the resulting matrix is TP. In the case of changing J_n to TP, the greatest number of 1's that can be achieved asymptotically is $O(n^{4/3})$. This means that the maximum number of equal entries in either a TP_2 or TP matrix is $\Theta(n^{4/3})$.

There are important, and previously unnoticed, connections between these ideas and a number of other mathematical concepts. As well as equal entries and minimally perturbable sets of entries, these include: incidences of points and lines in the Euclidean

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