Linear Algebra Appl. 454 (2014) 138–157

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Controllability of undirected graphs

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ARTICLE INFO

Article history:
Received 19 August 2013
Accepted 23 April 2014
Available online 13 May 2014
Submitted by H. Woerdeman

MSC:
05C50
05C22
05C38
05C90
34H05
37N35
47N70
93B05
93C15

Keywords:
Network dynamical systems
Controllable systems
Leaders and followers
Core vertices
Eigenvectors
Graph automorphisms
Omnicontrollable graphs

ABSTRACT

In control theory, networked dynamical systems have a wide range of engineering applications. In a relational graph among followers ($\mathbf{F}$) and leaders ($\mathbf{R}$), new necessary and sufficient conditions for a pair $(\mathbf{F}, \mathbf{R})$ to be controllable are presented. The choice of leader vertices for controllability is shown to be facilitated by identifying the core vertices associated with the eigenvectors of a matrix $\mathbf{S}$ related to a graph. We present new necessary and sufficient conditions for a graph to be controllable relative to its adjacency matrix or to its signless Laplacian without having to evaluate any eigenspaces, which is the criterion usually employed. The symmetries of the system graph represented by $\mathbf{S}$ are also shown to aid in the choice of a potential leader vertex that is able to control the follower subgraph on its own. Moreover, we define k-omnicontrollable graphs for controllability by any k leaders and show that simple 1-omnicontrollable graphs have only two possible automorphism groups.

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http://dx.doi.org/10.1016/j.laa.2014.04.022

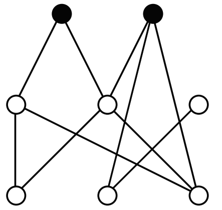

Fig. 1. An example of a networked dynamical system $G^\star$ with two leaders (the black vertices) and six followers (the white vertices).

1. Introduction

In this paper, we consider only undirected graphs with weighted edges and loops. If the weights of all the loops of an undirected graph G are zero and the edge weights of G are equal to one, then G is said to be *simple*. The vertex set of a graph G will be denoted by $\mathcal{V}(G)$, while the edge set of a simple graph H will be denoted by $\mathcal{E}(H)$.

Networked dynamical systems have lately attracted much research attention, both in the control and in the graph theory community. A networked dynamical system can be modelled by a graph G where the vertices represent agents (partitioned into f followers and ℓ leaders) and an edge $\{i, j\}$ is weighted by a_{ij} according to the strength of the exchange link between the two agents. Each follower agent is not just affected by the information exchanged with other followers, but also by the signals received from leaders that attempt to control the followers in order to direct their information to some predetermined state. Usually, there is only one leader agent, but systems requiring more than one leader agent are possible. Thus we essentially have two graphs: the entire graph $G^\star$ consisting of the leaders and followers with their interconnections and the subgraph G of $G^\star$ consisting only of the follower vertices with the edges joining them [11,13]. The graph $G^\star$ will be referred to as the *system graph*, while G will be called the *follower graph*, as shown in Fig. 1.

In a system of n follower agents with no leaders, the dynamics $\dot{x}_i$ of the state x_i of the ith vertex is taken to depend on those of its immediate neighbours, so that we obtain:

$$\dot{\mathbf{x}}(t) = \mathbf{S}\mathbf{x}(t) \tag{1}$$

where $\mathbf{x}(t)$ is a vector whose entries are the states x_i, $\dot{\mathbf{x}}(t)$ is a vector whose entries are the dynamics $\dot{x}_i$ and $\mathbf{S}$ is a real and symmetric matrix representing the relationship between the dynamics and the current states.

In the *adjacency matrix* $\mathbf{M}(G)$ representing the graph G, if i and j are two distinct vertices of G, then the ijth entry of $\mathbf{M}(G)$ is 0 if there is no edge connecting i and j and is the weight assigned to the edge (i, j) otherwise. The ith diagonal entry of $\mathbf{M}(G)$ represents the weight of a loop at vertex i. Thus $\mathbf{M}(G) = (a_{ij})$, so that $\mathbf{M}(G)$ is, in general, different from $\mathbf{S}$. It is usual to denote $\mathbf{M}(G)$ by $\mathbf{A}$ when G is simple. For