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On some restrictions of an operator to an invariant subspace

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For Banach spaces we consider the bounded linear operators which are surjective and noninjective. We show some general properties of such mappings. We examine whether such operators can be restricted to an involution or a projection. Thus, we will show that there exist many invariant subspaces for those operators. In respect to this, we will understand better the structure of many operators.

1. Introduction

Let X be a real or complex Banach space. Throughout this paper, the space X is assumed to be infinite-dimensional. The Banach space of all bounded linear operators from X to X is denoted by $\mathcal{B}(X)$. Now, we consider an operator $A \in \mathcal{B}(X)$ such that

$$A(X) = X \quad \text{and} \quad 0 < \dim \ker A = n. \tag{1.1}$$

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The operators satisfying (1.1) belong to the class of the Fredholm operators and their properties can be found, e.g., in [1, Chapter XI].

Since $\dim \ker A < \infty$, there is a closed subspace M such that $X = \ker A + M$ and $\ker A \cap M = \{0\}$. Since $A(X) = X$, we get that $A|_M : M \to X$ is a bijection. The spaces M, X are complete. Thus $(A|_M)^{-1} : X \to M$ is also continuous (by the Inverse Mapping Theorem).

Lemma 1.1. *Let X be a Banach space and let $T \in \mathcal{B}(X)$. Let an operator $A \in \mathcal{B}(X)$ have the following properties: $A(X) = X$, $0 < \dim \ker A = n$, $X = \ker A + M$ and $\ker A \cap M = \{0\}$, where M is a closed subspace of X. Suppose that $\|(A|_M)^{-1}\| \cdot \|T\| < 1$. Then there exists $B \in \mathcal{B}(X)$ such that $\dim B(\ker A) = \dim \ker A$ and $ABx = BTx$ for all $x \in \ker A$.*

Proof. Define $E := (A|_M)^{-1}$. Therefore $E : X \to M$ and $\|E\| \cdot \|T\| < 1$. Now, we define an operator $B := \sum_{k=0}^{\infty} E^k T^k$. Note that $\|E^k T^k\| \leqslant (\|E\| \cdot \|T\|)^k$. Since $\|E\| \cdot \|T\| < 1$, the series $\sum_{k=0}^{\infty} (\|E\| \cdot \|T\|)^k$ converges. Hence the series $\sum_{k=0}^{\infty} E^k T^k$ is absolutely convergent in $\mathcal{B}(X)$. Thus $B \in \mathcal{B}(X)$. Furthermore, it is easy to verify that $I + EBT = B$.

Let $y \in \ker A \setminus \{0\}$. Moreover $EBTy \in M$. Since $\ker A \cap M = \{0\}$ and $y + EBTy = By$, we get that $By \neq 0$. Thus, $B|_{\ker A} : \ker A \to X$ is injective, so $\dim B(\ker A) = \dim \ker A$. Finally, for every $x \in \ker A$, we have:

$$ABx = A(x + EBTx) = Ax + AEBTx = 0 + A(A|_M)^{-1} BTx = BTx$$

We have proved that $ABx = BTx$ for all $x \in \ker A$. $\square$

2. Main result

If $A \in \mathcal{B}(X)$ and L is a subspace of X, say that L is an *invariant subspace* for A if $A(L) \subset L$. Assume that (1.1) holds. In this paper we will prove that there exists a subspace $L \subset X$ such that $\alpha A|_L : L \to L$ is an involution (a projection or a nilpotent) for some $\alpha > 0$. It means that $A|_L$ is a scalar multiple of an involution (a projection or a nilpotent). It is clear that the subspace L will be the invariant subspace. Furthermore, we expect that the invariant subspace L will be nontrivial, i.e., $\{0\} \neq L \nsubseteq \ker A$ and $L \neq A(X)$.

An operator $T \in \mathcal{B}(X)$ is said to be *k-involution* (briefly *involution*), if $T^k = I$ for some positive integer k. An operator $P \in \mathcal{B}(X)$ is said to be *k-projection* (briefly *projection*) if $P^k = P$ for some positive integer k. An operator $N \in \mathcal{B}(X)$ is said to be *nilpotent* if $N^k = 0$ for some k.

Generally, the problem arises. Namely, given a polynomial p, do there exist $\alpha > 0$ and an invariant subspace L of dimension n such that $p(\alpha A|_L) = 0$? We will show that this problem has an affirmative answer.