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When is every matrix over a division ring a sum of an idempotent and a nilpotent?

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ABSTRACT

A ring is called nil-clean if each of its elements is a sum of an idempotent and a nilpotent. In response to a question of S. Breaz et al. in [1], we prove that the $n \times n$ matrix ring over a division ring D is a nil-clean ring if and only if $D \cong \mathbb{F}_2$. As consequences, it is shown that the $n \times n$ matrix ring over a strongly regular ring R is a nil-clean ring if and only if R is a Boolean ring, and that a semilocal ring R is nil-clean if and only if its Jacobson radical $J(R)$ is nil and $R/J(R)$ is a direct product of matrix rings over $\mathbb{F}_2$.

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1. Introduction

A ring is called nil-clean if each of its elements is a sum of an idempotent and a nilpotent. Nil-clean rings were extensively investigated by Diesl in [2] and [3]. Motivated by Diesl's question whether the matrix ring over a nil-clean ring is again nil-clean, S. Breaz et al. in [1] proved their main result that the matrix ring $\mathbb{M}_n(F)$ over a field F is nil-clean if and only if $F \cong \mathbb{F}_2$. This result has several interesting consequences, including a complete characterization of the finite rank Abelian groups with nil-clean endomorphism ring (see [1]). The proof of this result heavily relies on the commutativity of a field, and it was asked in [1] whether the result can be proved for a division ring instead of a field. As a response to this question, we prove that the matrix ring $\mathbb{M}_n(D)$ over a division ring D is a nil-clean ring if and only if $D \cong \mathbb{F}_2$. As consequences, it is shown that the matrix ring $\mathbb{M}_n(R)$ over a strongly regular ring R is a nil-clean ring if and only if R is a Boolean ring, and that a semilocal ring R is nil-clean if and only if its Jacobson radical $J(R)$ is nil and $R/J(R)$ is a direct product of matrix rings over $\mathbb{F}_2$ (as suggested in [1]).

Throughout, rings are associative with nonzero identity. The Jacobson radical of a ring R is denoted by $J(R)$. We write $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R, I_n for the $n \times n$ identity matrix, and $\mathbb{F}_2$ for the field of two elements.

2. The results

Our first lemma was proved in [3], which clearly implies that a nil-clean ring R with $J(R) = 0$ has characteristic 2.

Lemma 1. *(See [3, Proposition 3.14].) Let R be a nil-clean ring. Then the element 2 is a (central) nilpotent and, as such, is always contained in $J(R)$.*

Our second lemma is the main result in [1].

Lemma 2. *(See [1, Theorem 3].) Let F be a field and let $n \geqslant 1$. Then $\mathbb{M}_n(F)$ is a nil-clean ring if and only if $F \cong \mathbb{F}_2$.*

We are now ready to prove our main result.

Theorem 3. *Let D be a division ring and let $n \geqslant 1$. Then $\mathbb{M}_n(D)$ is a nil-clean ring if and only if $D \cong \mathbb{F}_2$.*

Proof. ($\Longleftarrow$). This is by Lemma 2.

($\Longrightarrow$). First note that $2 = 0$ in D by Lemma 1. Assume on the contrary that D is not isomorphic to $\mathbb{F}_2$. To get a contradiction, take $a \in D \backslash \{0, 1\}$. Then a, $1 - a$ are not nilpotents of D, and this implies that $a \in D$ is not a sum of an idempotent and a nilpotent. Hence the claim holds for $n = 1$. Let us assume that $n \geqslant 2$. By hypothesis,