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Permutation-like matrix groups with a maximal cycle of prime square length

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ABSTRACT

A matrix group is said to be permutation-like if any matrix of the group is similar to a permutation matrix. G. Cigler proved that, if a permutation-like matrix group contains a normal cyclic subgroup which is generated by a maximal cycle and the matrix dimension is a prime, then the group is similar to a permutation matrix group. This paper extends the result to the case where the matrix dimension is a square of a prime.

1. Introduction

A multiplicative group consisting of complex invertible matrices of size $n \times n$ is said to be a matrix group of dimension n. A matrix group $\mathcal{G}$ is said to be permutation-like if any matrix of $\mathcal{G}$ is similar to a permutation matrix, see [2,3]. If there exists an invertible matrix Q such that $Q^{-1}AQ$ is a permutation matrix for all $A \in \mathcal{G}$, then we say that $\mathcal{G}$

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is similar to a permutation matrix group, or $\mathcal{G}$ is a permutation matrix group for short. A matrix is called a maximal cycle if it is similar to a permutation matrix corresponding to a cycle permutation with cycle length equal to the dimension. G. Cigler in [3] showed that a permutation-like matrix group is not a permutation matrix group in general, and suggested a conjecture as follows.

Conjecture. *A permutation-like matrix group containing a maximal cycle is similar to a permutation matrix group.*

G. Cigler in [3] proved it affirmatively in two cases: the dimension $\leqslant 5$, or the dimension is a prime integer and the cyclic subgroup generated by the maximal cycle is normal.

In this paper we extend the result of [3] to the case where the length of the maximal cycle is a square of a prime.

Theorem 1.1. *Let $\mathcal{G}$ be a permutation-like matrix group of dimension p^2 where p is a prime. If $\mathcal{G}$ contains a maximal cycle C such that the subgroup $\langle C\rangle$ generated by C is normal in $\mathcal{G}$, then $\mathcal{G}$ is a permutation matrix group.*

In Section 2 we state some preliminaries as a preparation. The theorem will be proved in Section 3.

2. Preparation

The complex field is denoted by $\mathbb{C}$. For a positive integer n, by $\mathbb{Z}_n^*$ we denote the multiplicative group consisting of units of the residue ring $\mathbb{Z}_n$ of the integer ring $\mathbb{Z}$ modulo n. A diagonal blocked matrix

$$\begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix}$$

is denoted by $B_1 \oplus \cdots \oplus B_k$ for short. The identity matrix of dimension n is denoted by $I_{n\times n}$. All complex invertible matrices of dimension n form the so-called general linear group, denoted by $\mathrm{GL}_n(\mathbb{C})$. We denote the characteristic polynomial of a complex matrix A by $\mathrm{char}_A(x)$.

Lemma 2.1. *The following two are equivalent to each other:*

(i) *A is similar to a permutation matrix;*
(ii) *A is diagonalizable and* $\mathrm{char}_A(x) = \prod_i (x^{\ell_i} - 1)$.

If it is the case, then each factor $x^{\ell_i} - 1$ of $\mathrm{char}_A(x)$ corresponds to exactly one ℓ_i-cycle of the cycle decomposition of the permutation of the permutation matrix.