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Regularizing decompositions for matrix pencils and a topological classification of pairs of linear mappings

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ABSTRACT

By Kronecker's theorem, each matrix pencil $A+\lambda B$ over a field $\mathbb{F}$ is strictly equivalent to its *regularizing decomposition*; i.e., a direct sum

$$(I_r+\lambda D)\oplus(M_1+\lambda N_1)\oplus\cdots\oplus(M_t+\lambda N_t),$$

where D is an $r\times r$ nonsingular matrix and each $M_i+\lambda N_i$ is of the form $I_k+\lambda J_k(0)$, $J_k(0)+\lambda I_k$, $L_k+\lambda R_k$, or $L_k^T+\lambda R_k^T$, in which L_k and R_k are obtained from I_k by deleting its last or, respectively, first row and $J_k(0)$ is a singular Jordan block.

We give a method for constructing a regularizing decomposition of an $m\times n$ matrix pencil $A+\lambda B$, which is formulated in terms of the linear mappings $A,B:\mathbb{F}^n\to\mathbb{F}^m$.

Two $m\times n$ pencils $A+\lambda B$ and $A'+\lambda B'$ over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ are said to be *topologically equivalent* if the pairs of linear mappings $A,B:\mathbb{F}^n\to\mathbb{F}^m$ and $A',B':\mathbb{F}^n\to\mathbb{F}^m$ coincide up to homeomorphisms of the spaces $\mathbb{F}^n$ and $\mathbb{F}^m$. We prove that two pencils are topologically equivalent if and only if their regularizing decompositions coincide up to permutation of summands and replacement of D by a nonsingular matrix D' such that the linear operators $D,D':\mathbb{F}^r\to\mathbb{F}^r$ coincide up to a homeomorphism of $\mathbb{F}^r$.

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1. Introduction

In this article

- a regularizing decomposition of a matrix pencil is constructed by a method that is formulated in terms of images, preimages, and kernels of linear mappings, and
- the problem of topological classification of pairs of linear mappings $A, B : \mathbb{F}^n \to \mathbb{F}^m$ over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is reduced to the open problem of topological classification of linear operators, which was solved in special cases (in particular, for operators without eigenvalues that are roots of 1) in [4–9,12,13,18,21].

1.1. A regularizing decomposition of matrix pencils

A *matrix pencil* over a field $\mathbb{F}$ is a parameter matrix $A + \lambda B$, in which A and B are matrices over $\mathbb{F}$ of the same size. Two matrix pencils $A + \lambda B$ and $A' + \lambda B'$ are *strictly equivalent* if there exist nonsingular matrices S and R over $\mathbb{F}$ such that $S(A + \lambda B)R = A' + \lambda B'$. This means that the corresponding matrix pairs (A, B) and (A', B') are *equivalent*; i.e.,

$$SA = A'R \quad \text{and} \quad SB = B'R \quad \text{for some nonsingular } S \text{ and } R. \tag{1}$$

In what follows, we consider matrix pairs (A, B) instead of pencils $A + \lambda B$.

Denote by $J_k(0)$ the $k \times k$ singular Jordan block with units under the diagonal. Write

$$L_k := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad R_k := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \qquad \big((k-1)\text{-by-}k\big);$$

note that $L_1 = R_1 = 0_{01}$ is the 0×1 matrix of the linear mapping $\mathbb{F} \to 0$. Kronecker's canonical form for matrix pencils (see [11, Section XII]) ensures that each matrix pair (A, B) over a field $\mathbb{F}$ is equivalent to a direct sum

$$(I_r, D) \oplus (M_1, N_1) \oplus (M_2, N_2) \oplus \cdots \oplus (M_t, N_t) \tag{2}$$

in which D is an $r \times r$ nonsingular matrix and each (M_i, N_i) is one of the matrix pairs

$$\big(I_k, J_k(0)\big), \big(J_k(0), I_k\big), (L_k, R_k), \big(L_k^T, R_k^T\big), \quad k = 1, 2, \ldots \tag{3}$$