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A conjecture on the primitive degree of tensors ☆

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ABSTRACT

In this paper, we prove the following: Let $\mathbb{A}$ be a nonnegative primitive tensor with order m and dimension n. Then its primitive degree $\gamma(\mathbb{A}) \leqslant (n-1)^2+1$, and the upper bound is sharp. This confirms a conjecture of Shao [5].

1. Introduction

In [1] and [2], Chang et al. investigated the properties of the spectra of nonnegative tensors. They defined the irreducibility of tensors, and the primitivity of nonnegative

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tensors, and extended many important properties of primitive matrices of primitive tensors. Recently, as an application of the general tensor product defined by Shao [5], Shao presented a simple characterization of the primitive tensors in terms of the zero pattern of the powers of $\mathbb{A}$. He also proposed the following conjecture on the primitive degree $\gamma(\mathbb{A})$.

Conjecture 1.1. *When m is fixed, then there exists some polynomial $f(n)$ on n such that $\gamma(\mathbb{A}) \leqslant f(n)$ for all nonnegative primitive tensors of order m and dimension n.*

In this paper, we confirm the conjecture by proving the following theorem.

Theorem 1.2. *Let $\mathbb{A}$ be a nonnegative primitive tensor with order m and dimension n. Then its primitive degree $\gamma(\mathbb{A}) \leqslant (n-1)^2+1$, and the upper bound is sharp.*

2. Preliminaries

An order m and dimension n tensor $\mathbb{A} = (a_{i_1 i_2 \cdots i_m})_{1 \leqslant i_j \leqslant n\,(j=1,\ldots,m)}$ over the complex field $\mathbb{C}$ is a multidimensional array with all entries $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}$ $(i_1, \ldots, i_m \in [n] = \{1, \ldots, n\})$. The majorization matrix $M(\mathbb{A})$ of the tensor $\mathbb{A}$ is defined as $(M(\mathbb{A}))_{ij} = a_{ij\cdots j}(i, j \in [n])$ by Pearson [3].

Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geqslant 2$ (and $k \geqslant 1$), dimension n tensor, respectively. Recently, Shao [5] defined a general product $\mathbb{A}\mathbb{B}$ to be the following tensor $\mathbb{D}$ of order $(m-1)(k-1)+1$ and dimension n:

$$d_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} b_{i_2\alpha_1} \cdots b_{i_m\alpha_{m-1}} \quad \left(i \in [n],\ \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}\right).$$

The tensor product possesses a very useful property: the associative law [5, Theorem 1.1]. With the general product, Shao [5] proved some results on the primitivity and primitive degree of nonnegative tensors. The following result will be used in Definition 2.3.

Proposition 2.1. *(See [5, Proposition 4.1].) Let $\mathbb{A}$ be an order m and dimension n nonnegative tensor. Then the following three conditions are equivalent:*

(1). For any $i, j \in [n]$, $a_{ij\cdots j} > 0$ holds.

(2). For any $j \in [n]$, $\mathbb{A}e_j > 0$ holds (where e_j is the jth column of the identity matrix I_n).

(3). For any nonnegative nonzero vector $x \in \mathbb{R}^n$, $\mathbb{A}x > 0$ holds.

Definition 2.2. (See [3, Definition 3.1].) A nonnegative tensor $\mathbb{A}$ is called essentially positive, if it satisfies (3) of Proposition 2.1.

By Proposition 2.1, the following Definition 2.3 is equivalent to Definition 2.2.