Linear Algebra and its Applications 450 (2014) 202–216

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# Inverse eigenvalue problem of distance matrix via orthogonal matrix

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ARTICLE INFO

*Article history:*
Received 12 December 2012
Accepted 9 February 2014
Available online 27 March 2014
Submitted by R. Brualdi

*MSC:*
15A18
15A48
15A51

*Keywords:*
Distance matrix
Inverse eigenvalue problem

ABSTRACT

In this paper, for a given list $\sigma$ of real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, with sum zero and some additional more technical conditions specified in the paper, we construct a Euclidean distance matrix (EDM) having $\sigma$ as its list of eigenvalues, without using Hadamard matrices.



## 1. Introduction and preliminaries

A matrix $D = (d_{ij}) \in M_{n\times n}$ is said to be a Euclidean distance matrix (EDM), if there are $n$ points $x_1, x_2, \dots, x_n \in \mathbb{R}^r$, such that $d_{ij} = \|x_i - x_j\|^2$ for all $i, j = 1, 2, \dots, n$, where $\|\,.\,\|$ denotes the Euclidean norm. By this definition of EDM the following properties immediately hold.

1. $D$ is nonnegative matrix.

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http://dx.doi.org/10.1016/j.laa.2014.02.017

2. $D$ is symmetric.
3. $D$ has zero main diagonal and this means that the sum of its eigenvalues is zero.

(1.1)

If $x_1, x_2, \dots, x_n \in \mathbb{R}^r$ are the constructive points of EDM $D$, then $X = (x_1\ x_2\ \dots\ x_n)^T \in M_{n\times r}$ is called its coordinate matrix, where the $i$th row of this matrix is the coordinate of $x_i$. Since translation and rotation preserve the distance between two points, then coordinate matrix associated with an EDM is not unique. The minimum rank of the coordinate matrices associated with an EDM $D$ is called an embedding dimension of $D$ and denoted by $\mathrm{ed}(D)$. If $e$ is a vector with all elements 1, then $D$ is a distance matrix if and only if $D$ is negative semidefinite on $e^\perp = \{y \in \mathbb{R}^n, y^T e = 0\}$ [10]. Therefore an EDM $D$ has at most one positive eigenvalue on the set $e^\perp$ with dimension $n-1$. By attention to property (3) of (1.1) we conclude that $D$ has exactly one positive eigenvalue. Distance matrices are used in applications in geodesy, economics, genetics, psychology, biochemistry, engineering, etc.

Let $S_H$ denote the set of symmetric matrices of order $n$ with zero diagonal and let $S_C$ denote the set of symmetric matrices $A$ of order $n$ with $Ae = 0$. We define the following maps:

$$T: S_H \to S_C \quad \text{and} \quad K: S_C \to S_H$$

as

$$T(D) = -\frac{1}{2}\left(I - \frac{ee^T}{n}\right)D\left(I - \frac{ee^T}{n}\right),$$
$$K(B) = \mathrm{diag}(B)e^T + e\big(\mathrm{diag}(B)\big)^T - 2B.$$

The linear maps $T$ and $K$ are mutually inverse, and $D \in S_H$ is an EDM of embedding dimension $r$ if and only if $T(D)$ is positive semidefinite of rank $r$ [1].

An EDM $D$ is said to be spherical if the construction points of $D$ lie on a hypersphere, otherwise, it said to be non-spherical. By [2] we know that a distance matrix $D$ of embedding dimension $r$ is spherical if and only if its rank is $r+1$ and $D$ is non-spherical if and only if its rank is $r+2$. A spherical EDM $D$ is called regular if the constructive points of $D$ lie on a hypersphere whose center coincides with the centroid of those points. $D$ is regular spherical if and only if $e$ is the eigenvector of $D$ corresponding to the eigenvalue $\frac{e^T De}{n}$ [3]. If $D$ is regular spherical of embedding dimension $r$ and $A = T(D)$, then the $r$ negative eigenvalues of $D$ are exactly the eigenvalues of $-2A$ [4].

The characteristic polynomial of a spherical EDM $D$ is given by

$$P(\lambda) = \lambda^{n-r-1}\left[\left(\lambda - \frac{e^T De}{n}\right)\prod_{i=1}^{r}(\lambda - a_i) - \sum_{i=1}^{r} {b_i}^2 \prod_{j=1, j\neq i}^{r}(\lambda - a_j)\right] \tag{1.2}$$