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Linear maps characterized by the action on square-zero elements

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ABSTRACT

Let $R = \mathrm{M}_n(F)$, where $n \in \mathbb{N}$ and F is a field with char $F \neq 2$. We describe a linear map $f : R \to R$ with the property that $xf(x) = 0$ for all $x \in R$ with $x^2 = 0$.

1. Results

Let $R = \mathrm{M}_n(F)$, where $n \in \mathbb{N}$ and F is a field, and let $f : R \to R$ be an F-linear map. Under some mild conditions, Chebotar, Ke and Lee [1, Theorem 1] investigated such a map f that $f(x)f(y) = 0$ whenever $xy = 0$. As a related result, they [2, Theorem 4.1] also characterized a surjective linear map preserving square-zero elements in a Lie ideal of R.

Here we consider a similar situation with $xf(y)$ in place of $f(x)f(y)$. From a theorem due to Chuang and Lee [3, Theorem 2.3], it follows easily that if f satisfies $xf(y) = 0$

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for all $x, y \in R$ with $xy = 0$, then f is of the form $f(x) = xa$ for some $a \in R$. This result inspires us to investigate the condition that $xf(x) = 0$ whenever $x^2 = 0$.

Note that any square-zero element in R has zero trace. Therefore, if f assumes the form $f(x) = xa + \operatorname{tr}(x)b$ for some fixed $a, b \in R$, then $xf(x) = 0$ whenever $x^2 = 0$. We will show that this is the only case for an f with such a property that $xf(x) = 0$ whenever $x^2 = 0$, except when $n = 3$ or $\operatorname{char} F = 2$. A more precise statement of our result is as follows.

Theorem 2.3. *Let $R = \mathrm{M}_n(F)$, where $n \neq 3$ and F is a field with $\operatorname{char} F \neq 2$. Suppose that $f : R \to R$ is an F-linear map. Then the following are equivalent.*

(i) $xf(x) = 0$ for all $x \in R$ with $x^2 = 0$.
(ii) There exist $a, b \in R$ such that $f(x) = xa + \operatorname{tr}(x)b$ for all $x \in R$.

2. Proofs

We begin with the case $n = 2$.

Lemma 2.1. *Let $R = \mathrm{M}_2(F)$, where F is a field with $\operatorname{char} F \neq 2$. Suppose that $f : R \to R$ is an F-linear map such that $xf(x) = 0$ for all $x \in R$ with $x^2 = 0$. Then there exist $a, b \in R$ such that $f(x) = xa + \operatorname{tr}(x)b$ for all $x \in R$.*

Proof. Since $e_{12}^2 = 0$, it follows that $e_{12}f(e_{12}) = 0$. Analogously, $e_{21}f(e_{21}) = 0$.

Since $(e_{11} + e_{12} - e_{21} - e_{22})^2 = 0$, it follows that

$$\begin{aligned} 0 &= (e_{11} + e_{12} - e_{21} - e_{22})f(e_{11} + e_{12} - e_{21} - e_{22}) \\ &= (e_{11} + e_{12} - e_{21} - e_{22})\big(f(e_{11}) + f(e_{12}) - f(e_{21}) - f(e_{22})\big) \end{aligned}$$

and hence

$$\begin{aligned} 0 &= (e_{11} + e_{12})\big(f(e_{11}) + f(e_{12}) - f(e_{21}) - f(e_{22})\big) \\ &= e_{11}f(e_{11}) + e_{11}f(e_{12}) - e_{11}f(e_{22}) + e_{12}f(e_{11}) - e_{12}f(e_{21}) - e_{12}f(e_{22}). \end{aligned} \tag{2.1}$$

Analogously, $(e_{11} - e_{12} + e_{21} - e_{22})^2 = 0$ implies that

$$\begin{aligned} 0 &= (e_{11} - e_{12})\big(f(e_{11}) - f(e_{12}) + f(e_{21}) - f(e_{22})\big) \\ &= e_{11}f(e_{11}) - e_{11}f(e_{12}) - e_{11}f(e_{22}) - e_{12}f(e_{11}) - e_{12}f(e_{21}) + e_{12}f(e_{22}). \end{aligned} \tag{2.2}$$

Equating (2.1) and (2.2), we have

$$e_{11}f(e_{11}) = e_{11}f(e_{22}) + e_{12}f(e_{21})$$