Linear Algebra and its Applications 450 (2014) 250–279

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Classification of isometries of spaces of constant curvature and invariant subspaces

Joana Cirici

Fachbereich Mathematik und Informatik, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany

ARTICLE INFO

Article history:
Received 16 December 2013
Accepted 6 March 2014
Available online 29 March 2014
Submitted by J.M. Landsberg

MSC:
15A21
51M10
51F25

Keywords:
Isometries
Normal forms
Segre symbol
Orbit types
z-Classes
Invariant subspaces
Orthogonal group
Euclidean group
Lorentz group
Hyperbolic space

ABSTRACT

We study the varieties of invariant totally geodesic submanifolds of isometries of the spherical, Euclidean and hyperbolic spaces in each finite dimension. We show that the dimensions of the connected components of these varieties determine the orbit type (or the z-class) of the isometry. For this purpose, we introduce the Segre symbol of an isometry, a discrete invariant encoding the structure of its normal form, which parametrizes z-classes. We then provide a description of the isomorphism type of the varieties of invariant subspaces in terms of the Segre symbol.

1. Introduction

Our objective is to study the isometries of spaces of constant curvature. We will denote by $\mathbf{I}(\mathbb{M}^n)$ the isometry group of $\mathbb{M}^n$, where $\mathbb{M}^n$ denotes one of the simply connected space forms of finite dimension n: the sphere $\mathbb{S}^n$, the Euclidean space $\mathbb{E}^n$ or the hyperbolic space $\mathbb{H}^n$.

http://dx.doi.org/10.1016/j.laa.2014.03.008
0024-3795/

A very first and well known classification of elements of $\mathbf{I}(\mathbb{M}^n)$ is based on their fixed point behavior. The *translation length* of an isometry f is the infimum of the distances that points are moved. If f has a fixed point it is called *elliptic*. If the translation length is positive then f is called *hyperbolic*. If f does not fix a point but its translation length is zero, then it is called *parabolic*. We remark that the latter case can only occur in hyperbolic spaces (see for example [1]). This is a very coarse classification, only useful up to a certain extent, and in some sense analogous to the distinction between direct and indirect isometries.

At the other extreme, we can consider the classification of isometries by conjugacy classes. This gives infinitely many classes, each one represented by a normal form. A natural step is to identify those classes differing in the sets of eigenvalues but having the same normal form structure. This leads to what we call the *Segre decomposition*: a decomposition of $\mathbf{I}(\mathbb{M}^n)$ into finitely many classes, each an uncountable union of conjugacy classes, parametrized by a discrete invariant.

The *Segre symbol* of a linear endomorphism of a finite dimensional complex vector space is a sequence of positive integers encoding the number of distinct eigenvalues, together with the sizes of the Jordan blocks corresponding to each eigenvalue of a Jordan normal form of the endomorphism. This invariant has been studied by many authors for classification purposes, such as the treatment of collineations and pencils of quadrics developed in §.VIII of [2] or Petrov's classification of gravitational fields appearing in §.3 of [3], in which the Segre symbol of the curvature tensor allows to stratify Einstein spaces.

Arnold [4] studied the conjugation action of $\mathrm{GL}(n,\mathbb{C})$ on the space $\mathcal{M}_n(\mathbb{C})$ of complex square matrices of size n and suggested that the partition into Segre classes (defined by those matrices with a given Segre symbol) makes $\mathcal{M}_n(\mathbb{C})$ into a stratified space. Gibson [5] proved this statement and showed that the stratification is Whitney-regular. Furthermore, Broer [6] showed that in this case, the partition according to the Segre symbol coincides with the stratification by orbit types.

Consider a Lie group G acting on a manifold M. Then two elements of M are said to have the same *orbit type* if their orbits are G-equivariantly isomorphic. This is equivalent to the condition that their isotropy subgroups are conjugate. In the case in which a group acts on itself by conjugation, then the orbit types are also called *centralizer classes* or *z-classes*. This notion was introduced by Kulkarni in [7] in his study of "dynamical types". The classification by z-classes using characteristic and minimal polynomials has been developed in many particular cases, such as for the general linear and affine groups [8], the anisotropic groups of type G_2 defined over a field [9], or the groups of isometries of the real, quaternionic and complex hyperbolic spaces [10–12].

In the present paper we define the Segre symbol of isometries of $\mathbb{M}^n$ using normal forms. Our definition is a natural adaptation from the original Segre symbol of a linear endomorphism, to isometries of spaces of constant curvature, and involves the distinction between elliptic, hyperbolic and parabolic isometries. We then study the z-classes of $\mathbf{I}(\mathbb{M}^n)$ and show that the Segre decomposition coincides with the decomposition by z-classes. In particular, the number of z-classes is finite, and we can count them by