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# Comparison of the quasi-inverses of the Kronecker sum and product of matrices over complete commutative dioids with applications

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ABSTRACT

We compare the quasi-inverses of the Kronecker sum $\boxplus$ and product $\boxtimes$ of two given square matrices $A$ and $B$, with entries from an idempotent, complete and commutative dioid. We prove that $(A \boxplus B)^*$ is greater than or equal to $(A \boxtimes B)^*$ in the sense of the canonical order, where $^*$ denotes the quasi-inverse. We also show how to reduce the computational complexity of computing $(A \boxplus B)^*$ from sixtic to quartic order. Moreover, we propose three applications of our results. First, we compare solutions to fixed-point type matrix equations $X = A \otimes X \otimes B \oplus C$ and $X = A \otimes X \oplus X \otimes B \oplus C$ over any complete commutative dioid. It is also possible to compare the connections between vertices in the corresponding graphs $G(A \boxplus B)$ and $G(A \boxtimes B)$ over the Boolean dioid. Finally, we show that the lengths of the shortest paths in the aforementioned graphs are equal using the tropical dioid.

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## 1. Preliminaries

**Definition 1** *(Dioid).* (See [1, p. 154].) A dioid is a set $\mathcal{D}$ endowed with two operations denoted by $\oplus$ and $\otimes$ (called 'sum' or 'addition' and 'product' or 'multiplication') satisfying the following properties:

1. associativity of addition;
2. commutativity of addition;
3. associativity of multiplication;
4. distributivity of multiplication with respect to addition;
5. existence of a zero element, i.e., $\exists \varepsilon \in \mathcal{D}$: $\forall a \in \mathcal{D}$, $a \oplus \varepsilon = a$;
6. absorbing zero element, i.e., $\forall a \in \mathcal{D}$, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$;
7. existence of an identity element, i.e., $\exists e \in \mathcal{D}$: $\forall a \in \mathcal{D}$, $a \otimes e = e \otimes a = a$; and
8. idempotency of addition, i.e., $\forall a \in \mathcal{D}$, $a \oplus a = a$.

Note that the definition of a dioid in [2, p. 28] does not include the last property above. However, in this paper a dioid is always idempotent.

**Definition 2** *(Commutative dioid).* (See [1, p. 155].) A dioid is commutative if it is commutative with respect to multiplication.

Throughout this paper we consider commutative dioids. We will denote $\underbrace{a \otimes \cdots \otimes a}_{k \text{ times}}$ by $a^k$, $k \in \mathbb{N}$, and $a^0 = e$.

An order relation is a binary relation (denoted by $\geqslant$) that is reflexive, transitive and antisymmetric, and the order is total if each pair of elements is comparable; otherwise, the order is partial. Also, a set endowed with a total or partial order relation is a totally or partially ordered set, respectively.

**Theorem 1** *(Canonical order relation). (See [1, p. 160], [2, pp. 12–13].) In a dioid $\mathcal{D}$, one has the following equivalence*:

$$\forall a, b\colon\ a = a \oplus b \quad \Leftrightarrow \quad \exists c\colon\ a = b \oplus c.$$

*Moreover, these equivalent statements define a (partial) order relation denoted by $\geqslant$ as follows*:

$$a \geqslant b \quad \Leftrightarrow \quad a = a \oplus b. \tag{1}$$

*This order relation is compatible with addition, that is to say,*

$$a \geqslant b \quad \Rightarrow \quad \{\forall c,\ a \oplus c \geqslant b \oplus c\},$$

*and multiplication, that is,*

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