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Spectra of a class of non-self-adjoint matrices

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ABSTRACT

We consider a new class of non-self-adjoint matrices that arise from an indefinite self-adjoint linear pencil of matrices, and obtain the spectral asymptotics of the spectra as the size of the matrices diverges to infinity. We prove that the spectrum is qualitatively different when a certain parameter c equals 0, and when it is non-zero, and that certain features of the spectrum depend on Diophantine properties of c.

1. Introduction

The spectral theory of pencils of linear operators has a long history, with contributions by distinguished people including Krein, Langer, Gohberg, Pontryagin and Shkalikov. It

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has many applications, for example to control theory, mathematical physics and vibrating structures. We refer to [8] and [14] for accounts of this subject and extensive bibliographies. Among the theoretical tools that have been developed to study some such problems is the theory of Krein spaces, which also has a long history, [7] and [4]. There is also a substantial numerical literature on self-adjoint linear and quadratic pencils, [10,11,5]. An interesting physically motivated example with some unusual features has recently been considered in [6].

It is well-known that self-adjoint pencils may have complex eigenvalues. If the pencil depends on a real parameter c in addition to the spectral parameter, which we always call λ, one often sees two real eigenvalues of the pencil meeting at a square root singularity as c changes, and then emerging as a complex conjugate pair, or vice versa. However, little has been written about the distribution of the complex eigenvalues, and the unexpected phenomena revealed in this paper show how hard a full understanding is likely to be. Some of these phenomena may disappear when studying suitable infinite-dimensional pencils of differential operators, but, if so, the reason for this will need to be explained.

The paper studies the simplest, non-trivial example of a finite-dimensional, self-adjoint, linear pencil, and in particular the asymptotic distribution of its non-real eigenvalues as the dimension increases. Our main results are presented in Theorem 4.2, which is illustrated in Fig. 2, and in Theorem 5.4, illustrated in Fig. 6. However, a number of other cases exhibiting further complexities are also considered. Numerical studies indicate that some of the phenomena described here occur for a much larger class of pencils, including the case in which the matrix D defined in (2) is indefinite and has slowly varying coefficients on each of the two subintervals concerned. Proving this is a task for the future.

A self-adjoint, linear pencil of $N \times N$ matrices is defined to be a family of matrices of the form $\mathcal{A} = \mathcal{A}(\lambda) = H - \lambda D$, where H, D are self-adjoint $N \times N$ matrices and $\lambda \in \mathbb{C}$. The number λ_0 is said to be an *eigenvalue* of this pencil $\mathcal{A}$ if $\mathcal{A}(\lambda_0) = H - \lambda_0 D$ is not invertible, or equivalently if $Hv = \lambda_0 Dv$ has a non-zero solution $v \in \mathbb{C}^N$. The *spectrum* of the pencil $\mathcal{A}$ is the set of all its eigenvalues, and will be denoted by $\operatorname{Spec}(\mathcal{A})$. It coincides with the set of all roots of the polynomial

$$p(\lambda) = \det(H - \lambda D).$$

We always assume that D is invertible, so that $p(\lambda)$ is a polynomial of degree N with non-zero leading coefficient. The spectrum of the pencil equals that of the matrix $D^{-1}H$, which is generically non-self-adjoint. In the standard case when D is the identity matrix, the spectrum of the pencil $H - \lambda I$ coincides with the spectrum of the matrix H, which we also denote by $\operatorname{Spec}(H)$.

The spectrum of such a self-adjoint pencil is real if either H or D is a definite matrix, i.e. all of its eigenvalues have the same sign. If both H and D are sign-indefinite matrices the problem is said to be indefinite, and it is known that the spectrum may then be