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The inertia of weighted unicyclic graphs ☆

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A B S T R A C T

Let G_w be a weighted graph. The *inertia* of G_w is the triple $In(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w)$, $i_-(G_w)$, $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of the adjacency matrix $A(G_w)$ of G_w including their multiplicities, respectively. $i_+(G_w)$, $i_-(G_w)$ are called the *positive, negative indices of inertia* of G_w, respectively. In this paper we present a lower bound for the positive, negative indices of weighted unicyclic graphs of order n with fixed girth and characterize all weighted unicyclic graphs attaining this lower bound. Moreover, we characterize the weighted unicyclic graphs of order n with two positive, two negative and at least $n - 6$ zero eigenvalues, respectively.

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1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* $A(G) = (a_{ij})$ of graph G of order n is a symmetric $(0,1)$-matrix such that $a_{ij} = 1$ if v_i is adjacent to v_j and 0 otherwise. A weighted graph G_w is a pair (G, w) where G is a simple graph with edge set $E(G)$, called the *underlying graph* of G_w, and w is a weight function from $E(G)$ to the set of nonzero real numbers. The *adjacency matrix* of G_w on n vertices is defined as the matrix $A(G_w) = (a_{ij})$ such that $a_{ij} = w(v_iv_j)$ if v_i is adjacent to v_j and 0 otherwise. The characteristic polynomial of G_w is the characteristic polynomial of $A(G_w)$, denoted by

$$P_{G_w}(\lambda) = \det\bigl(\lambda I - A(G_w)\bigr) = \lambda^n + a_1^*\lambda^{n-1} + \cdots + a_n^*.$$

The *inertia* of G_w is defined to be the triple $In(G_w) = (i_+(G_w), i_-(G_w), i_0(G_w))$, where $i_+(G_w)$, $i_-(G_w)$, $i_0(G_w)$ are the numbers of the positive, negative and zero eigenvalues of $A(G_w)$ including multiplicities, respectively. $i_+(G_w)$ and $i_-(G_w)$ are called the *positive, negative indices of inertia* (abbreviated *positive, negative indices*) of G_w, respectively. The number $i_0(G_w)$ is called the *nullity* of G_w. The rank of an n-vertex graph G_w, denoted by $r(G_w)$, is defined as the rank of $A(G_w)$. Obviously, $r(G_w) = i_+(G_w) + i_-(G_w) = n - i_0(G_w)$.

A graph G_w is called *acyclic* (resp. *unicyclic*, *bipartite*) if its underlying graph G is *acyclic* (resp. *unicyclic*, *bipartite*). An *induced subgraph* of G_w is an induced subgraph of G with the same weights. For a subgraph H_w of G_w, let $G_w - H_w$ be the subgraph obtained from G_w by deleting all vertices of H_w and all incident edges. For $V' \subseteq V(G_w)$, $G_w - V'$ is the subgraph obtained from G_w by deleting all vertices in V' and all their incident edges. A vertex of a graph G_w is called *pendant* if it has degree one, and is called *quasi-pendant* if it is adjacent to a pendant vertex. For a weighted graph G_w on at least two vertices, a vertex $v \in V(G_w)$ is called *unsaturated* in G_w if there exists a maximum matching M of G in which no edge is incident with v; otherwise, v is called *saturated* in G_w.

A simple graph may be regarded as a weighted graph in which the weight of each edge is $+1$. A signed graph may be regarded as a weighted graph in which the weight of each edge is $+1$ or -1. Moreover, the sign of a signed cycle, denoted by $sgn(C)$, is defined as the sign of the product of all edge weights $+1$ or -1 on C. The signed cycle C is said to be *positive* (*or negative*) if $sgn(C) = +$ (or $sgn(C) = -$). A signed graph is said to be *balanced* if all its cycles are positive, otherwise it is called *unbalanced*.

The study of eigenvalues of a weighted graph has attracted much attention. Several results about the (Laplacian) spectral radius of weighted graphs were derived in [1,10,9,24,25]. The inertia of unweighted graphs has attracted some attention. Gregory et al. [17] studied the subadditivity of the positive, negative indices of inertia and developed certain properties of Hermitian rank which were used to characterize the biclique decomposition number. Gregory et al. [16] investigated the inertia of a partial join of two graphs and