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On the extension of some total positivity inequalities ☆

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ABSTRACT

Some inequalities recently obtained for totally positive matrices are extended to more general classes of matrices. Among these larger classes of matrices, sign consistent, sign regular and matrices with signed bidiagonal decomposition are considered. The class of matrices with a signed bidiagonal decomposition contains the nonsingular totally positive matrices and their inverses.

1. Introduction

The extension of properties satisfied by totally positive matrices to more general classes of matrices is, in general, a difficult problem. This paper explores the extension of some inequalities recently proved. Let us recall that a matrix is *totally positive* (TP)

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if all its minors are nonnegative. TP matrices are also called in the literature totally nonnegative matrices. Applications of these matrices can be seen in the references of [1] and of the books [4] and [11]. Among the classes of matrices considered in this paper, we can mention the well-known sign consistent matrices of given order (see [6,8]), the sign regular matrices (see [6,9]) and matrices with a signed bidiagonal decomposition. The class of matrices with a signed bidiagonal decomposition has been introduced in [2] and contains matrices very important in applications, such as nonsingular totally positive matrices or their inverses. In [2] it was shown that many computations (such as the calculation of singular values, eigenvalues or the inverses) with matrices with a signed bidiagonal decomposition can be performed with high relative accuracy.

Section 2 extends to the more general classes mentioned previously the lower bound for the minimal eigenvalue of a nonsingular totally positive matrix obtained in [10]. The Hadamard product of nonsingular Jacobi TP matrices is also a nonsingular Jacobi TP matrix and, in Section 3, we prove that the Hadamard product of nonsingular Jacobi matrices with a signed bidiagonal decomposition also inherits this property, although a similar result does not hold for nonsingular Jacobi SR matrices. The Schur complement plays an important role in many fields. It is well known (cf. [4]) that Schur complements of invertible principal submatrices of TP matrices with contiguous index sets are also TP, although the result is not valid for all index sets. In contrast, we prove in Theorem 3.6 that Schur complements of invertible principal submatrices of matrices with signed bidiagonal decomposition always belong to this class of matrices. Finally, we also extend some Schur complement inequalities for the Hadamard product satisfied by totally positive matrices to the class of matrices with signed bidiagonal decomposition.

2. Inequalities for the minimal eigenvalue

We denote by λ_* an eigenvalue with minimal absolute value of a given matrix A; that is, $|\lambda_*| \leqslant |\lambda_i|$ for all eigenvalues λ_i of A. Let us recall (cf. [1, Corollary 6.6]) that a nonsingular TP matrix has positive eigenvalues. Therefore, if A is a nonsingular TP matrix, then we can take $\lambda_* > 0$. In [10], a bound for this λ_* was obtained for nonsingular TP matrices, improving the bound given by the well-known Gerschgorin Theorem for the localization of eigenvalues. Here we extend the bound for more classes of matrices.

Given $k, l \in \{1, 2, \dots, n\}$, let α (resp., β) be any increasing sequence of k (resp., l) positive integers less than or equal to n. Let A be a real square matrix of order n. Then we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. Besides we denote a principal submatrix by $A[\alpha] := A[\alpha|\alpha]$. The *complement* α^c is the increasingly rearranged $\{1, 2, \dots, n\} \setminus \alpha$. We shall denote $A(\alpha|\beta] := A[\alpha^c|\beta]$, $A[\alpha|\beta) := A[\alpha|\beta^c]$, $A(\alpha|\beta) := A[\alpha^c|\beta^c]$ and $A(\alpha) := A[\alpha^c]$. Given $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$, we define the matrix $|A| := (|a_{ij}|)_{1 \leqslant i,j \leqslant n}$.

Given an $n \times n$ real matrix A and $i \leqslant n$, we say that A is *sign consistent* of order i, SC_i, with sign $\varepsilon_i \in \{\pm 1\}$ if all minors of order i have the same sign ε_i (not necessarily strict). Observe that TP matrices are SC_i with sign $+1$ for all i.