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The minimum rank of a sign pattern matrix with a 1-separation

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A sign pattern matrix is a matrix whose entries are from the set $\{+, -, 0\}$. If A is an $m \times n$ sign pattern matrix, the qualitative class of A, denoted $Q(A)$, is the set of all real $m \times n$ matrices $B = [b_{i,j}]$ with $b_{i,j}$ positive (respectively, negative, zero) if $a_{i,j}$ is $+$ (respectively, $-$, 0). The minimum rank of a sign pattern matrix A, denoted $\mathrm{mr}(A)$, is the minimum of the ranks of the real matrices in $Q(A)$. Determination of the minimum rank of a sign pattern matrix is a longstanding open problem.

For the case that the sign pattern matrix has a 1-separation, we present a formula to compute the minimum rank of a sign pattern matrix using the minimum ranks of certain generalized sign pattern matrices associated with the 1-separation.

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0. Introduction

A *sign pattern matrix* (or *sign pattern*) is a matrix whose entries are from the set $\{+,-,0\}$. If $B=[b_{i,j}]$ is a real matrix, then $\operatorname{sgn}(B)$ is the sign pattern matrix $A=[a_{i,j}]$ with $a_{i,j}=+$ (respectively, $-$, 0) if $b_{i,j}$ is positive (respectively, negative, zero). If A is a sign pattern matrix, the *sign pattern class* of A, denoted $Q(A)$, is the set of all real matrices $B=[b_{i,j}]$ with $\operatorname{sgn}(B)=A$. The *minimum rank* of a sign pattern matrix A, denoted $\operatorname{mr}(A)$, is the minimum of the ranks of matrices in $Q(A)$; see [5]. Recently, Li et al. [7] obtained a characterization of sign pattern matrices A with $\operatorname{mr}(A)\leqslant 2$. In this paper, we present a formula to compute the minimum rank of a sign pattern matrix with a 1-separation using the minimum ranks of certain generalized sign pattern matrices associated with the 1-separation.

The notion of sign pattern matrix can be extended to generalized sign pattern matrices by allowing certain entries to be #; see [5]. For a generalized sign pattern matrix A, the generalized sign pattern class of A, denoted $Q(A)$, is defined by allowing entries of a matrix $B=[b_{i,j}]\in Q(A)$ to be any real number if the corresponding entries of A are #. The minimum rank $\operatorname{mr}(A)$ of a generalized sign pattern matrix A is defined in the same way as for a sign pattern matrix: $\operatorname{mr}(A)$ is the minimum of the ranks of matrices in $Q(A)$. If $A=[a_{i,j}]$ and $C=[c_{i,j}]$ are generalized sign pattern matrices of the same size, we write $A\leqslant C$ if for each entry of A, $a_{i,j}=c_{i,j}$ or $c_{i,j}=\#$. It is clear that if $A\leqslant C$, then $Q(A)\subseteq Q(C)$. For a generalized sign pattern C, let $\mathcal{C}$ be the set of all sign pattern matrices A such that $A\leqslant C$. Then, clearly, $Q(C)=\bigcup_{A\in\mathcal{C}}Q(A)$. Hence the minimum rank of a generalized sign pattern matrix C equals $\min_{A\in\mathcal{C}}\operatorname{mr}(A)$.

We define subtraction of two elements from $\{+,-,0\}$ as follows:

1. $(+)-(0)=+$, $(0)-(-)=+$, $(+)-(-)=+$,
2. $(-)-(+)=-$, $(0)-(+)=-$, $(-)-(0)=-$,
3. $(0)-(0)=0$,
4. $(+)-(+)=\#$, $(-)-(-)=\#$.

The idea behind the definition of, for example, $(-)-(+)=-$ is that subtracting a positive number from a negative number gives a negative number.

Let

$$M=\begin{bmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & a_{2,2} & A_{2,3} \\ 0 & A_{3,2} & A_{3,3} \end{bmatrix}$$

be a sign pattern matrix, where $A_{1,2}$ and $A_{3,2}$ have only one column, and $A_{2,1}$ and $A_{2,3}$ have only one row. We also say that the sign pattern matrix M has a 1-*separation*. For $p\in\{+,-,0\}$, let