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The signature of line graphs and power trees ☆

Long Wang, Yi-Zheng Fan *

School of Mathematics Sciences, Anhui University, Hefei 230601, PR China

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ABSTRACT

Let G be a graph and let $A(G)$ be the adjacency matrix of G. The signature $s(G)$ of G is the difference between the positive inertia index and the negative inertia index of $A(G)$. Ma et al. (2013) [10] conjectured that $-c_3(G) \leqslant s(G) \leqslant c_5(G)$, where $c_3(G)$ and $c_5(G)$ respectively denote the number of cycles in G which have length $4k+3$ and $4k+5$ for some integers $k \geqslant 0$, and proved the conjecture holds for trees, unicyclic or bicyclic graphs.

It is known that $s(G) = 0$ if G is bipartite, and the signature is closely related to the odd cycles or nonbipartiteness of a graph from the existing results. In this paper we show that the conjecture holds for line graphs and power trees.

1. Introduction

Throughout this paper we consider only simple graphs. The *adjacency matrix* $A(G) = [a_{ij}]$ of a graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$ is defined to be a symmetric matrix of order n such that $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The *positive inertia index* $p(G)$, the *negative inertia index* $n(G)$ and the *nullity* $\eta(G)$ of G are respectively defined to be the number of positive eigenvalues,

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* Corresponding author.
E-mail addresses: wanglongxuzhou@126.com (L. Wang), fanyz@ahu.edu.cn (Y.-Z. Fan).

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negative eigenvalues and zero eigenvalues of $A(G)$. The *rank* of G, written as $r(G)$, is defined to be the rank of $A(G)$. The *signature* of G, denoted by $s(G)$, is defined to be the difference $p(G) - n(G)$. Obviously, $p(G) + n(G) + \eta(G) = |V(G)|$, $p(G) + n(G) = r(G)$ and $p(G) - n(G) = s(G)$.

Motivated by the discovery that the nullity of a graph is related to the stability of the molecular represented by the graph [1] and the open problem of characterizing all singular graphs posed by Collatz [2], many authors discussed the nullity of a graph and obtained a significant number of interesting results. Here we particularly mention the results involved with the nullity of line graphs. In 1998, Sciriha [12] showed that the order of every tree whose line graph is singular is even and also that the nullity of the line graph of a tree is at most one. A new proof of the latter result by Gutman and Sciriha appeared later in [6]. Li et al. [8] proved that the nullity of the line graph of a unicyclic graph with depth one is at most two. Gong and Xu [5] improved the above results. They showed that the nullity of the line graph of a connected graph with k induced cycles is at most $k + 1$.

Recently some authors discuss a more general problem, that is, describing the positive or negative inertia index of graphs or weighted graphs, especially of trees or their line graphs, unicyclic or bicyclic graphs; see Ma et al. [10,11], Li and Song [9] and Yu et al. [13,14]. In the paper [10] the authors posed a conjecture as follows, and proved the conjecture holds for trees, unicyclic or bicyclic graphs.

Conjecture 1.1. *(See [10].) The inequality $-c_3(G) \leqslant s(G) \leqslant c_5(G)$ possibly holds for any simple graph G, where $c_3(G)$ and $c_5(G)$ denote respectively the number of cycles having length $4k + 3$ (or length 3 modulo 4) and the number of cycles having length $4k + 5$ for some integers $k \geqslant 0$ (or length 1 modulo 4).*

Theorem 1.2. *(See [10].) Let G be a tree, or a unicyclic graph, or a bicyclic graph. Then $-c_3(G) \leqslant s(G) \leqslant c_5(G)$.*

A weaker result was also given by Ma et al. [10] that $|s(G)| \leqslant c_1(G)$ for any graph G, where $c_1(G)$ denotes the number of odd cycles of G, that is $c_1(G) = c_3(G) + c_5(G)$.

When G is bipartite, surely $s(G) = 0$ and the conjecture holds in this case. So, from Theorem 1.2 or Conjecture 1.1 (if it were true), we find that the signature is closely related to the odd cycles or nonbipartiteness of a graph. In this paper we prove that the conjecture holds for line graphs and power trees.

2. Preliminaries

We first introduce some notation. Let G be a graph and let $W \subseteq V(G)$. Denote by $G - W$ the subgraph of G obtained by deleting the vertices in W together with all edges incident to them. If G_1 is a subgraph of G, we sometimes write $G - G_1$ instead of $G - V(G_1)$. In particular, if $W = \{x\}$, we simply write $G - W$ as $G - x$. If G_1 is an