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An optimal test for almost strict total positivity

J.M. Peña [1]

Departamento de Matemática Aplicada/IUMA, Universidad de Zaragoza, Spain

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ABSTRACT

A matrix is almost strictly totally positive if all its minors are nonnegative and they are positive if and only if they do not contain a zero in their diagonal. An optimal test to check if a given matrix belongs to this class of matrices is presented. For this purpose, we establish a bijection between the set of nonzero entries of the matrix and a set of submatrices called essential submatrices, which are explicitly constructed. The test shows that it is sufficient to check the positivity of the essential minors, improving the characterization presented in [15]. Essential minors are also applied to the construction of accurate bidiagonal decompositions of almost strictly totally positive matrices, which in turn can be used for deriving accurate algorithms for these matrices.

1. Introduction

A real matrix is called totally positive (TP) if all its minors are nonnegative, and it is called strictly totally positive (STP) if they are positive. Matrices with all minors nonnegative (in particular all positive) have been applied to many fields of mathematics and their applications, including Approximation Theory, Differential

E-mail address: jmpena@unizar.es.
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Equations, Statistics, Computer Aided Geometric Design, Quantum Groups, Combinatorics and Economics (cf. [17,1,10,21,9]). Let us mention that TP and STP matrices are also called totally nonnegative and totally positive matrices, respectively.

A class of matrices intermediate between TP and STP matrices is the class of matrices called *almost strictly totally positive* (ASTP) in [11]. ASTP matrices are matrices whose minors are positive if and only if all their diagonal entries are positive (see also [21]). Nonsingular ASTP matrices form the most interesting class of TP matrices for some important applications. For instance, collocation matrices of B-splines [3] and Hurwitz matrices [2,7,18] provide examples of nonsingular ASTP matrices, and they can be also applied to important problems of interpolation [5]. Nonsingular ASTP matrices were called in [16] *inner totally positive matrices.*

The problem of finding tests for the recognition of TP and STP matrices has been one of the earliest problems considered in this field (see the surveys [10] and [20]). From the beginning of the study of STP matrices it has been known that it is not necessary to check the sign of all minors of a matrix to decide whether or not it is STP. The optimal criterion to check if a matrix is STP was obtained in Theorem 4.1 of [12]. It only requires n^2 minors to check if an $n \times n$ matrix is STP. In [15], the boundary submatrices of nonsingular ASTP matrices were introduced and we proved in Theorem 2.4 that the positivity of the boundary minors is sufficient to check if a matrix is nonsingular ASTP, improving previous characterizations appeared in the literature. Here we improve in Theorem 2 the previous criterion extending the optimal test of STP matrices to nonsingular ASTP matrices: we prove that if a square matrix A has k nonzero entries, then k minors (which we call essential minors) are sufficient to check if A is nonsingular ASTP. The essential minors of a nonsingular ASTP matrix are explicitly defined in Definition 2.6 and they form a subclass of the boundary minors.

Bidiagonal factorizations of TP matrices and related matrices have also played a key role in this subject (see [14,8]). We also use the essential minors to construct explicitly the bidiagonal factorization of nonsingular ASTP matrices. As a consequence, the relationship of essential boundary minors with the construction of accurate algorithms for almost strictly totally positive matrices is analyzed.

Section 2 introduces the essential minors and proves the optimal determinantal test for nonsingular ASTP matrices. Among other illustrative examples to apply the test, an example with Hurwitz matrices is included. Section 3 relates the essential minors with the bidiagonal factorization of nonsingular ASTP matrices. A practical interest of bidiagonal factorizations comes from the fact that an accurate bidiagonal factorization of a nonsingular TP matrix leads to many accurate computations, such as the calculation of singular values and eigenvalues (see [19]). In Theorem 5 we construct explicitly a bidiagonal decomposition of a nonsingular ASTP matrix from its essential minors through a subtraction-free (and so, accurate) procedure.