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Complete multipartite graphs are determined by their distance spectra ☆

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A B S T R A C T

In this paper, we prove that the complete multipartite graphs are determined by their distance spectra, which confirms the conjecture proposed by Lin, Hong, Wang and Shu (2013) [7], although it is well known that the complete multipartite graphs cannot be determined by their adjacency spectra.

1. Introduction

In this paper, we only consider simple and undirected connected graphs. Let $G = (V(G), E(G))$ be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. Let $d_G(v)$ and $N_G(v)$ denote the degree and neighbors of a vertex v, respectively. $D(G) = (a_{uv})_{n\times n}$

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denotes the *distance matrix* of G with $a_{uv} = d_G(u, v)$, where $d_G(u, v)$ is the distance between vertices u and v. The largest eigenvalue of $D(G)$, denoted by $\lambda(G)$, is called the *distance spectral radius* of G. The research for distance matrix can be dated back to the paper [6,5], which presents an interesting result that the determinant of the distance matrix of trees with order n is always $(-1)^{n-1}(n-1)2^{n-2}$, independent of the structure of the tree. Recently, the distance matrix of a graph has received increasing attention. For example, Liu [8] characterized the graphs with minimal spectral radius of the distance matrix in three classes of simple connected graphs with fixed vertex connectivity, matching number and chromatic number, respectively. Zhang [10] determined the unique graph with minimum distance spectral radius among all connected graphs with a given diameter. Bose, Nath and Paul [1] characterized the graph with minimal distance spectral radius among all graphs with the fixed number of pendent vertices.

Denote by $Sp(D(G))$ the set of all eigenvalues of $D(G)$ including the multiplicity. Two graphs G and G' are called *D-cospectral* if $Sp(D(G)) = Sp(D(G'))$. A graph G is determined by its D-spectra if $Sp(D(G)) = Sp(D(G'))$ implies $G \cong G'$. There are three excellent surveys [2–4] on that which graphs can be determined by their spectra. Lin et al. [7] characterized all the connected graphs with smallest eigenvalue being -2 and proposed the following conjecture:

Conjecture 1.1. *(See [7].) Let $G = K_{n_1,n_2,\dots,n_k}$ be a complete k-partite graph. Then G is determined by its D-spectrum.*

Moreover, they proved that the conjecture is true for $k = 2$. On the other hand, it is well known that complete multipartite graphs cannot be determined by their spectra. For example, $K_{1,4}$ is not determined by its adjacency spectra, since $K_{1,4}$ and the union of cycle of order 4 and an isolated vertex have the same spectrum but are not isomorphic. In this paper, we will give a positive answer to Conjecture 1.1.

2. Main results

Before presenting the proof of the conjecture, we need the following theorems and lemmas.

Theorem 2.1. *(See [7].) Let $D(G)$ be the distance matrix of a connected graph G. Then the smallest eigenvalue of $D(G)$ is equal to -2 with multiplicity $n - k$ if and only if G is a complete k-partite graph for $2 \leqslant k \leqslant n - 1$.*

Theorem 2.2. *(See [7].) Let $G = K_{n_1,\dots,n_k}$ be a complete k-partite graph. Then the characteristic polynomial of $D(G)$ is*

$$P_D(\lambda) = (\lambda+2)^{n-k}\left[\prod_{i=1}^{k}(\lambda - n_i + 2) - \sum_{i=1}^{k} n_i \prod_{j=1,\, j\neq i}^{k}(\lambda - n_j + 2)\right].$$