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Lie algebras and bilinear forms in characteristic 2

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ABSTRACT

All bilinear forms defined on a finite dimensional vector space over an algebraically closed field of characteristic 2 whose associated Lie algebra is reductive are determined.

1. Introduction

Let V be a vector space of finite dimension $n \geqslant 1$ over a field F and let $\mathfrak{gl}(V)$ stand for the general linear Lie algebra of all endomorphisms of V under the bracket $[x, y] = xy - yx$. To every bilinear form $f : V \times V \to F$ there corresponds a subalgebra $L(f)$ of $\mathfrak{gl}(V)$, given by

$$L(f) = \{x \in \mathfrak{gl}(V) \mid f(xu, v) + f(u, xv) = 0 \text{ for all } u, v \in V\}.$$

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The complex symplectic and orthogonal Lie algebras, which correspond to non-degenerate alternating and symplectic bilinear forms, are known to be simple but for a few low dimensional exceptions [8]. Being simple or semisimple are such strong conditions that if we consider the Lie algebra of a totally arbitrary bilinear form over an arbitrary field no new cases of simplicity or semisimplicity arise [1].

A related but more flexible notion is reductivity. Recall that a Lie algebra is reductive if all its solvable ideals are central. All bilinear forms over an algebraically closed field of characteristic not 2 whose associated Lie algebra is reductive were determined in [1].

The goal of this paper is to solve this problem in characteristic 2, when additional obstacles arise. In order to state our main result, some matrix information will be required.

Corresponding to any $A \in \mathfrak{gl}(n)$, we have the subalgebra $L(A)$ of $\mathfrak{gl}(n)$, given by

$$L(A) = \{X \in \mathfrak{gl}(n) \mid X'A + AX = 0\},$$

where X' denotes the transpose of X.

Let $f : V \times V \to F$ be a bilinear form and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of V. The Gram matrix $A \in M_n(F)$ of f relative to $\mathcal{B}$ is defined by $A_{ij} = f(v_i, v_j)$. We say that A represents f with respect to $\mathcal{B}$. Given $x \in \mathfrak{gl}(V)$, let $M_{\mathcal{B}}(x)$ be the matrix of x relative to $\mathcal{B}$. Then $x \mapsto M_{\mathcal{B}}(x)$ is a Lie isomorphism between $\mathfrak{gl}(V)$ and $\mathfrak{gl}(n)$ mapping $L(f)$ onto $L(A)$.

Following [7], we define the matrices Γ_m, for odd m, as follows:

$$\Gamma_1 = (1), \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \ldots$$

Given any $m \geqslant 1$ and $\lambda \in F$, let $J_m(\lambda)$ stand for the lower triangular λ-Jordan block, and write J_m for $J_m(0)$. Thus

$$J_1(\lambda) = (\lambda), \qquad J_2(\lambda) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \qquad J_3(\lambda) = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \quad \ldots$$

Moreover, if $0 \neq \lambda \in F$, we consider the matrix

$$\Omega_m(\lambda) = \begin{pmatrix} 0 & J_m(\lambda) \\ I_m & 0 \end{pmatrix}.$$

Note that $\Omega_m(\lambda)$ and $\Omega_m(\lambda^{-1})$ represent the same bilinear form.

Theorem 1.1. *Suppose that F is an algebraically closed field of characteristic 2 and let $f : V \times V \to F$ be an arbitrary bilinear form defined on an F-vector space of finite*