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Bounds on elementary symmetric functions

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ABSTRACT

Tight bounds on an elementary symmetric function are established under the assumption that the values of the elementary symmetric functions of lower orders are given. The explicit form of the inverse Hankel moment matrix leads to inequalities for moments and for elementary symmetric polynomials.

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1. Setting of the problem

The paper concerns a bound that arises in a statistical meta-analysis model [12]. Specifically, let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a given set of real numbers, and let $E_m = E_m(\mathcal{S})$ be the m-th elementary symmetric polynomial in $s_1, \ldots, s_n$; i.e., E_m is the coefficient of x^{n-m} in the polynomial $P(x) = \prod_1^n (x + s_i)$. The problem in [12] (in the case of positive distinct s_i's) consists in obtaining tight bounds on E_n if all other elementary symmetric

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functions $E_1, \dots, E_{n-1}$ are given. We consider the somewhat more general problem of obtaining bounds on E_k, $2 \leqslant k \leqslant n$, for fixed values $E_1, \dots, E_{k-1}$.

An equivalent formulation of this problem is as follows. Let the monic polynomial $P(x)$ have only real roots $-s_1, \dots, -s_n$. Given the roots $-t_1, \dots, -t_{n-1}$ of the derivative,

$$P'(x) = \frac{d}{dx} \prod_1^n (x + s_i) = n \prod_1^{n-1} (x + t_j),$$

establish the range of possible values $P(0) = E_n(\mathcal{S})$. Since for $0 \leqslant k \leqslant n-1$, $(n-k)E_k(\mathcal{S}) = nE_k(\{t_1, \dots, t_{n-1}\})$, elementary symmetric functions of $t_1, \dots, t_{n-1}$ determine $E_1 = E_1(\mathcal{S}), \dots, E_{n-1} = E_{n-1}(\mathcal{S})$.

There are some general results on the extremal values of linear combinations of elementary symmetric functions over the real variety $\{E_1 = e_1, \dots, E_{k-1} = e_{k-1}\}$ [7]. For example each of the points where E_k attains local extrema has at most k different coordinates.

Our approach is based on the well known *Newton identities* [6] relating the elementary symmetric functions to the classical power sums,

$$M_k = \sum_i s_i^k, \quad k = 1, \dots, n.$$

Indeed the functions $E_1, \dots, E_k$ define $M_1, \dots, M_k$ and vice versa. For example,

$$\begin{aligned} M_k &= \det \begin{pmatrix} E_1 & 1 & 0 & \cdots & 0 \\ 2E_2 & E_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ kE_k & E_{k-1} & E_{k-2} & \cdots & E_1 \end{pmatrix} \\ &= \sum_{j_1 + 2j_2 + \cdots + kj_k = k} \nu_{j_1 j_2 \cdots j_k}^{(k)} E_1^{j_1} \cdots E_k^{j_k}, \end{aligned} \tag{1}$$

with integer coefficients $\nu_{j_1 j_2 \cdots j_k}^{(k)}$. In particular, $\nu_{00\cdots01}^{(k)} = (-1)^{k+1}k$.

Thus our problem reduces to that of specifying the range of M_k for the given values $M_1, \dots, M_{k-1}$. The latter problem has a known solution given in terms of canonical moments [3, Section 1.4]. Indeed the theory of canonical moments provides bounds on the k-th moment of a measure on an interval $[a, b]$ when the first $k-1$ moments are given.

The next section takes advantage of positive definiteness of Hankel matrices and provides the inequalities for moments and for elementary symmetric polynomials. Some examples are discussed in Section 3. A lower bound for weighted moments is given in Section 4. Numerical comparison with the known inequalities in Section 5 concludes this paper.