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Extended-valued topical and anti-topical functions on semimodules

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A B S T R A C T

In the papers [16] and [17] we have studied functions defined on a b-complete idempotent semimodule X over a b-complete idempotent semifield $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$, with values in $\mathcal{K}$, where $\mathcal{K}$ may (or may not) contain a greatest element $\sup \mathcal{K}$, and the residuation x/y is not defined for $x \in X$ and $y = \inf X$. In the present paper we assume that $\mathcal{K}$ has no greatest element, then adjoin to $\mathcal{K}$ an outside "greatest element" $\top = \sup \mathcal{K}$ and extend the operations $\oplus$ and $\otimes$ from $\mathcal{K}$ to $\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}$, so as to obtain a meaning also for $x/\inf X$, for any $x \in X$, and study functions with values in $\overline{\mathcal{K}}$. In fact we consider two different extensions of the product $\otimes$ from $\mathcal{K}$ to $\overline{\mathcal{K}}$, denoted by $\otimes$ and $\dot{\otimes}$ respectively, and use them to give characterizations of topical (i.e. increasing homogeneous, defined with the aid of $\otimes$) and anti-topical (i.e. decreasing anti-homogeneous, defined with the aid of $\dot{\otimes}$) functions in terms of some inequalities. Next we introduce and study for functions $f : X \to \overline{\mathcal{K}}$ their conjugates and biconjugates of Fenchel–Moreau type with respect to the coupling functions $\varphi(x, y) = x/y$, $\forall x, y \in X$, and $\psi(x, (y, d)) := \inf\{x/y, d\}$, $\forall x, y \in X$, $\forall d \in \overline{\mathcal{K}}$, and use them to obtain characterizations of topical and anti-topical functions. In the subsequent sections we consider for the coupling functions φ and ψ some concepts that have been studied in Rubinov and Singer (2001) [11] and Singer (2004) [15] for the so-called "additive min-type coupling functions" $\pi_\mu : R^n_{\max} \times R^n_{\max} \to R_{\max}$ and $\pi_\mu : A^n \times A^n \to A$ respectively, where A is a conditionally complete lattice ordered group and $\pi_\mu(x, y) := \inf_{1 \leqslant i \leqslant n}(x_i + y_i)$, $\forall x, y \in R^n_{\max}$ (or A^n). Thus, we

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study the polars of a set $G \subseteq X$ for the coupling functions φ and ψ, and we consider for a function $f : X \to \overline{\mathcal{K}}$ the notion of support set of f with respect to the set $\widetilde{\mathcal{T}}$ of all "elementary topical functions" $\widetilde{t}_y(x) := x/y, \forall x \in X, \forall y \in X \backslash \{\inf X\}$ and two concepts of support set of f at a point $x_0 \in X$. The main differences between the properties of the conjugations with respect to the coupling functions φ, ψ and π_μ and between the properties of the polars of a set G with respect to the coupling functions φ, ψ and π_μ are caused by the fact that while π_μ is symmetric, with values only in $R_{\max}$ (resp. A), φ and ψ are not symmetric and take values also outside $R_{\max}$ (resp. A).

1. Introduction

In the previous papers [16] and [17], attempting to contribute to the construction of a theory of functional analysis and convex analysis in semimodules over semifields, we have studied topical functions $f : X \to \mathcal{K}$ and related classes of functions, where X is a b-complete idempotent semimodule over a b-complete idempotent semifield $\mathcal{K}$. We recall that $f : X \to \mathcal{K}$ is called *topical* if it is *increasing* (i.e., the relations $x', x'' \in X, x' \leqslant x''$ imply $f(x') \leqslant f(x'')$, where $\leqslant$ denotes the canonical order on $\mathcal{K}$, respectively on X, defined by $\lambda \leqslant \mu \Leftrightarrow \lambda \oplus \mu = \mu$, $\forall \lambda \in \mathcal{K}$, $\forall \mu \in \mathcal{K}$, respectively by $x \leqslant y \Leftrightarrow x \oplus y = y$, $\forall x \in X$, $\forall y \in X$), and *homogeneous* (i.e., $f(\lambda x) = \lambda f(x)$ for all $x \in X, \lambda \in \mathcal{K}$, where $\lambda x := \lambda \otimes x, \lambda f(x) = \lambda \otimes f(x)$; the fact that we use the same notations for addition $\oplus$ both in $\mathcal{K}$ and in X and for multiplication $\otimes$ both in $\mathcal{K}$ and in $\mathcal{K} \times X$ will lead to no confusion). These definitions will be used also when $\mathcal{K}$ is replaced by $R = ((-\infty, +\infty), \oplus = \max$, $\otimes = +)$ although it is not a semiring, and X is replaced by R^n. Let us also recall that an idempotent semiring $\mathcal{K}$, that is, a semiring with idempotent addition $\oplus$ (i.e. such that $\lambda \oplus \lambda = \lambda$ for all $\lambda \in \mathcal{K}$) or an idempotent semimodule X (over an idempotent semiring $\mathcal{K}$) is called *b-complete*, if it is closed under the sum $\oplus$ of any subset (order-) bounded from above and the multiplication $\otimes$ distributes over such sums.

As in [16] and [17], we shall make the following *basic assumptions*:

(A0′) $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$ is a *b-complete idempotent semifield* (i.e., a b-complete idempotent semiring in which every $\mu \in \mathcal{K} \backslash \{\varepsilon\}$ is invertible for the multiplication $\otimes$, where ε denotes the neutral element of $(\mathcal{K}, \oplus)$), and the supremum of each (order-) bounded from above subset of $\mathcal{K}$ belongs to $\mathcal{K}$; also, X is a *b-complete idempotent semimodule* over $\mathcal{K}$. In the sequel we shall omit the word "idempotent"; this will lead to no confusion.

(A1) For all elements $x \in X$ and $y \in X \backslash \{\inf X\}$ the set $\{\lambda \in \mathcal{K} \mid \lambda y \leqslant x\}$ is (order-) bounded from above, where $\leqslant$ denotes the canonical order on $\mathcal{K}$, respectively on X.

Remark 1. a) It is easy to see that *an idempotent semifield $\mathcal{K}$ has no greatest element* $\sup \mathcal{K}$, *unless $\mathcal{K} = \{\varepsilon\}$ or $\mathcal{K} = \{\varepsilon, e\}$, where ε and e denote the neutral elements of* $(\mathcal{K}, \oplus)$