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Matrix division with an idempotent divisor or quotient

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A B S T R A C T

For given matrices F and G over an arbitrary field $\mathbb{F}$, necessary and sufficient conditions (in terms of rank, amongst others) are presented for F to divide G, where either F itself is idempotent or one of its quotients is idempotent, or both. Formulae are also given by which these quotients can be constructed. Finally, new proofs of some known results on idempotent factorization are presented in the present context to provide another perspective on the behaviour of such factorizations.

1. Introduction

A matrix $G \in M_{m\times n}(\mathbb{F})$, $\mathbb{F}$ an arbitrary field, is right divisible by $F \in M_{k\times n}(\mathbb{F})$ if $G = HF$ for some $H \in M_{m\times k}(\mathbb{F})$. F is called a right divisor (or right factor) of G, and H is called a right quotient of G and F (the remainder is always assumed to be zero). Similar definitions hold for left division.

Throughout, all matrices are assumed to be over an arbitrary fixed field $\mathbb{F}$, unless stated otherwise. G will denote an $m \times n$ matrix. The order of F will either be $m \times k$

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or $k \times n$, depending on whether it is a left or right divisor, but this will be clear from the context. The same applies for H. Of course, if F or H is idempotent, then it is also square.

This paper presents necessary and sufficient conditions (in terms of rank, amongst others) for F to be a divisor of G, where either F itself is idempotent or one of its quotients is idempotent, or both. Since the similarity class of an idempotent matrix is completely determined by its rank, it is useful to also investigate which ranks the idempotent quotients can have. Formulae are also given by which quotients with every possible rank can be constructed. Finally, new proofs of some known results on idempotent factorization, developed in [3,1,2], are presented in the present context to provide another perspective on the behaviour of such factorizations.

For a given matrix $G \in M_{m\times n}(\mathbb{F})$, it will sometimes be useful to consider the linear mapping from $\mathbb{F}^n$ to $\mathbb{F}^m$ associated with G with respect to the standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$. This is also denoted by G (hence, $G(v) = Gv$ for all $v \in \mathbb{F}^n$), since the intended meaning will be clear from the context. We denote the kernel (equivalently, null space) of G by $N(G)$, the image (equivalently, range or column space) of G by $R(G)$, and their respective dimensions by $n(G)$ and $r(G)$. The row space of G is denoted by $row(G)$. Similarity in $M_n(\mathbb{F})$ is denoted by $\approx$. A left (resp., right) inverse of $G \in M_{m\times n}(\mathbb{F})$, if it exists, is denoted by G^L (resp., G^R). If $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, then the conjugate transpose of G is denoted by G^* (note that $G^* = G^T$ if $\mathbb{F} = \mathbb{R}$).

For related research on the factorization of matrices over rings, such as the ring of integers, the reader is referred to [5], and references therein.

2. Preliminary results

The following two results are true for matrix division in general. The first provides a simple criterion for a matrix F to divide a matrix G, and establishes bounds for the ranks that the quotients can have. The second provides a formula by which the quotients of G and F can be generated, given that G is divisible by F.

Proposition 1. *Let $G \in M_{m\times n}(\mathbb{F})$ and let $F \in M_{k\times n}(\mathbb{F})$ (resp., $F \in M_{m\times k}(\mathbb{F})$), where $\mathbb{F}$ is a field. The following conditions are equivalent*:

(a) *There exists a matrix $H \in M_{m\times k}(\mathbb{F})$ (resp., $H \in M_{k\times n}(\mathbb{F})$) such that $G = HF$ (resp., $G = FH$);*

(b) $row(G) \subseteq row(F)$ $\quad$ $\big(resp.,\ R(G) \subseteq R(F)\big)$;

(c) $N(F) \subseteq N(G)$ $\quad$ $\big(resp.,\ N(F^T) \subseteq N(G^T)\big)$;

(d) $$r(F) = r\left(\begin{bmatrix} G \\ F \end{bmatrix}\right) \quad \big(resp.,\ r(F) = r([G \quad F])\big). \tag{1}$$