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A characterization of bipartite distance-regular graphs

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ABSTRACT

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular. Examples are given to show that the converse does not hold. Thus, a natural question is to find out when the converse is true. In this paper we give a quasi-spectral characterization of a connected bipartite weighted 2-punctually distance-regular graph whose halved graphs are distance-regular. In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

1. Introduction

The study of characterizing the graphs whose eigenvalues and/or multiplicities satisfy a prescribed identity has a long history. For example, a well-known and real-world applicable theory asserts that a connected graph is bipartite if and only if its largest eigenvalue and smallest eigenvalue have the same absolute value. Recently, the eigenvectors, especially the one associated with the largest eigenvalue, are also taking into consider-

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ation, for instances, in mathematical theory: [18,19,15,16,13,22]; in applications: [7,4]. See [6, pp. 65–69] for more applications. In this paper, we will give a (quasi-spectral) characterization of graphs when an identity involving eigenvalues, multiplicities, the eigenvector corresponding to the largest eigenvalue, and partial graph structure is satisfied. The details are as follows.

Throughout this paper, let G be a connected graph with vertex set V, order $n = |V|$, diameter D, and distance function ∂. The *adjacency matrix* A of G is the binary matrix indexed by V, where the entry $(A)_{uv} = 1$ if $\partial(u, v) = 1$, and $(A)_{uv} = 0$ otherwise. Assume that A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ with corresponding multiplicities $m_0 = 1, m_1, \ldots, m_d$. The *spectrum* of G is denoted by $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, and the parameter d is called the *spectral diameter* of G. Note that $D \leqslant d$ [3]. As is known, there is a sequence of orthogonal polynomials $p_0, p_1, \ldots, p_d$ with respect to the inner product $\langle\, ,\rangle_G$ (formally defined in the beginning of the next section), where $\deg p_i = i$ and $\langle p_i, p_i\rangle_G = p_i(\lambda_0)$ for $0 \leqslant i \leqslant d$ [15]. Let α be the eigenvector of A associated with λ_0 such that $\alpha^t\alpha = n$ and all entries of α are positive. Note that α is usually called the *Perron vector*, and $\alpha = (1, 1, \ldots, 1)^t$ if and only if G is regular. For $u \in V$, let α_u be the entry corresponding to u in the eigenvector α. For $0 \leqslant i \leqslant d$, define the *weighted distance-i matrix* $\widetilde{A}_i$ of G to be the matrix indexed by V such that the entry $(\widetilde{A}_i)_{uv} = \alpha_u\alpha_v$ if $\partial(u, v) = i$, and $(\widetilde{A}_i)_{uv} = 0$ otherwise. In particular, for the case G is regular, $\widetilde{A}_i$ is binary and is the so-called *distance-i matrix* A_i of G. For an integer $h \leqslant d$, we say that G is *weighted h-punctually distance-regular* if $\widetilde{A}_h = p_h(A)$. Define $\widetilde{\delta}_i = \sum_{u,v}(\widetilde{A}_i \circ \widetilde{A}_i)_{uv}/n$, where "$\circ$" is the entrywise product of matrices. A bipartite graph with bipartition (X, Y) is called (k_1, k_2)-*biregular* if every vertex in X has degree k_1 and every vertex in Y has degree k_2. The *distance-i graph* of G is the graph whose adjacency matrix is the distance-i matrix of G. For a connected bipartite graph G with bipartition (X, Y), the *halved graphs* G^X and G^Y are the two connected components of the distance-2 graph of G. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular [5, Proposition 4.2.2]. Examples 5.1–5.3 show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Thus, a natural question is to find out when the converse is true. Our main result is the following.

Theorem 1.1. *Let G be a connected bipartite graph with bipartition (X, Y). Suppose that G is weighted 2-punctually distance-regular with even spectral diameter, and both halved graphs G^X and G^Y are distance-regular. Then G is distance-regular.*

In addition to the main result, we believe that Proposition 3.3, Theorem 3.4, Proposition 4.5 and Theorem 5.6 are of independent interest.

This paper is organized as follows. In the next section we provide some simple but useful lemmas for bipartite graphs. In Section 3, we present some results related to the spectral excess theorem [15], and characterize the graphs with $\widetilde{\delta}_i = p_i(\lambda_0)$ for $i \in \{0, 1\}$ (Lemma 3.7). In particular, this lemma is very useful for checking the reg-