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On products of matrices of a fixed order

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ABSTRACT

We study products of matrices of fixed orders. We show that if g is an upper triangular matrix, finite or infinite, over a field of q elements, then g can be expressed as a product of at most four triangular matrices whose orders are divisors of $q - 1$. This result can be applied to the general linear and to the Vershik–Kerov group. We also present some facts about conjugacy of elements of orders dividing $q - 1$.

1. Introduction

One of the first notions that one learns about studying the group theory is the set of generators. In particular, it is known (see for instance [10]) that the group of all invertible matrices over a field K is generated by the matrices of forms

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$$\begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_n \end{pmatrix}, \quad \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \alpha \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad d_1, d_2, \ldots, d_n, \alpha \in K^*.$$

However, other sets of generators of the general linear group [9] and its subgroups [2,15] are also of interest. It should be mentioned that finding generators of $\mathrm{GL}_n(R)$ where R is any associative ring with 1 is far more difficult [20,27].

What we would like to focus on is the relation between the generators and the orders of elements in matrix groups. For the case of the group $\pm\mathrm{SL}_n(K)$, i.e. consisting of all matrices over K with determinant equal to 1 or -1, it was proved the following.

Theorem 1.1. *(See [25].) Every square matrix over a field, with determinant* ± 1*, is the product of not more than four involutions.*

There are some more known facts about products of involutions in groups of matrices over fields [4,5] and rings [8,12]. Some of them [22] concern matrices of infinite dimension.

One may ask whether any analogous result can be obtained for the elements whose orders are fixed but different from 2. We will investigate this issue in groups of upper triangular matrices. The matrices under consideration may be of finite dimension or infinite dimensional. We will show how we can express some of them as a products of matrices whose orders are finite.

It can be easily counted that if $|K| = q$, then $\mathrm{T}_n(K)$ – the group of $n \times n$ upper triangular matrices over K, consists of $(q-1)^n q^{\frac{n^2-n}{2}}$ elements and it contains a lot of matrices whose orders are coprime to $q-1$. These are the unitriangular matrices which form a Sylow p-subgroup of $\mathrm{T}_n(K)$, but also a Sylow p-subgroup of $\mathrm{GL}_n(K)$ (for some more information on this topic see [26]). Nevertheless, we will prove that matrices of order $q-1$ (and their divisors) play a special role in matrix groups. More precisely, we will show that we have

Theorem 1.2. *Let K be a field of q elements, $q > 2$, and let G be either a group $\mathrm{T}_\infty(K)$ or $\mathrm{T}_n(K)$ for some $n \in \mathbb{N}$. Then G is generated by the set of all matrices of orders $q-1$. Moreover, every element of G is a product of at most four triangular matrices whose orders are divisors of $q-1$.*

From results for triangular matrix groups we will be able to derive some information about generators of $\mathrm{GL}_n(K)$ – the general linear group. We will prove

Theorem 1.3. *Let K be a field of q elements, $q > 2$, and let $n \in \mathbb{N}$. Then $\mathrm{GL}_n(K)$ is generated by matrices of order $q-1$.*

More precisely, every matrix from $\mathrm{GL}_n(K)$ can be written as a product of at most twelve (upper or lower) triangular matrices whose orders divide $q-1$.