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On a conjecture for the signless Laplacian eigenvalues ☆

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A B S T R A C T

Let G be a simple graph with n vertices and $e(G)$ edges, and $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G) \geqslant 0$ be the signless Laplacian eigenvalues of G. Let $S_k^+(G) = \sum_{i=1}^k q_i(G)$, where $k = 1, 2, \ldots, n$. F. Ashraf et al. conjectured that $S_k^+(G) \leqslant e(G) + \binom{k+1}{2}$ for $k = 1, 2, \ldots, n$. In this paper, we give various upper bounds for $S_k^+(G)$, and prove that this conjecture is true for the following cases: connected graph with sufficiently large k, unicyclic graphs and bicyclic graphs for all k, and tricyclic graphs when $k \neq 3$. Finally, we discuss whether the upper bound given in this conjecture is tight or not for c-cyclic graphs and propose some problems for future research.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Laplacian matrix and the signless Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$ and

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$Q(G) = D(G) + A(G)$ respectively, where $A(G)$ is the adjacent matrix and $D(G)$ is the diagonal matrix of vertex degrees of G. It is well known that both $L(G)$ and $Q(G)$ are symmetric and positive semidefinite, then we can denote the eigenvalues of $L(G)$ and $Q(G)$, called respectively the Laplacian eigenvalues and the signless Laplacian eigenvalues of G, by $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$ and $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G) \geqslant 0$.

Let $|U|$ be the cardinality of a finite set U, and $|E(G)| = e(G)$. If $e(G) = n + c - 1$, then G is called a c-cyclic graph. Especially, if $c = 0$ ($c = 1$, $c = 2$, $c = 3$), then G is called an acyclic (a unicyclic, a bicyclic, a tricyclic) graph.

Grone and Merris [10] conjectured that for a graph with n vertices and degree sequence $\{d_v \mid v \in V(G)\}$, the following holds:

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G) \leqslant \sum_{i=1}^{k} \bigl|\{v \in V(G) \mid d_v \geqslant i\}\bigr|, \quad \text{for } k = 1, 2, \ldots, n.$$

Recently, it was proved by Bai [2]. As a variation of the Grone–Merris conjecture, Brouwer [4] conjectured that for a graph G with n vertices,

$$S_k(G) \leqslant e(G) + \binom{k+1}{2}, \quad \text{for } k = 1, 2, \ldots, n.$$

This conjecture attracted many researchers. Haemers et al. [11] showed that it is true for $k = 2$. Moreover, they obtained an upper bound of $S_k(T)$ for any tree T with n vertices and $k = 1, 2, \ldots, n$, i.e.,

$$S_k(T) \leqslant e(T) + 2k - 1 \leqslant e(T) + \binom{k+1}{2},$$

and proved it holds for trees and threshold graphs. This was improved in [8] to the stronger equality as follows:

$$S_k(T) \leqslant e(T) + 2k - 1 - \frac{2k-2}{n}, \quad \text{for } k = 1, 2, \ldots, n.$$

Moreover, the conjecture was proved to be true for unicyclic graphs, bicyclic graphs, regular graphs, split graphs, cographs and graphs with at most ten vertices. For more details, we refer readers to Refs. [3,4,6,11,13,14].

Motivated by the definition of $S_k(G)$ and Brouwer's conjecture, F. Ashraf et al. [1] proposed the following conjecture about $S_k^+(G)$, where $S_k^+(G) = \sum_{i=1}^{k} q_i(G)$ for $k = 1, 2, \ldots, n$.

Conjecture 1.1. *For any graph G with n vertices and any $k = 1, 2, \ldots, n$,*

$$S_k^+(G) \leqslant e(G) + \binom{k+1}{2}.$$