Linear Algebra and its Applications 446 (2014) 166–176

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Spectral moments of regular graphs in terms of subgraph counts

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ARTICLE INFO

Article history:
Received 14 August 2013
Accepted 19 December 2013
Available online 30 January 2014
Submitted by B.L. Shader

MSC:
05C50
15A18

Keywords:
Spectral moment
Regular graph
Adjacency matrix
Eigenvalue
Totally-reducible walk
Tree-like walk

ABSTRACT

We give equations that relate the spectral moments of a regular graph G to the numbers of certain subgraphs that occur within G. The equations are derived by using generating functions to count certain closed walks in G. The subgraphs that determine the i-th spectral moment are precisely the connected graphs of minimum degree 2 that are induced by at least one closed walk of length i.

1. Introduction

Our aim is to provide equations that give the spectral moments of a regular graph as a linear combination of the number of copies of certain small subgraphs of the graph.

All graphs in this paper are simple and undirected. Throughout, G will be an $(r+1)$-regular graph on n vertices. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the

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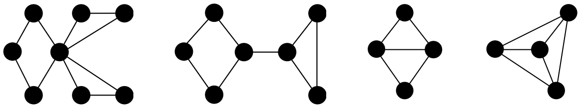
0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.12.028



(a) $C_{4\cdot 3\cdot 3}$ (b) C_{4-3} (c) $\Theta_{2,2,1}$ (d) K_4

Fig. 1. Contributor notation.

adjacency matrix of G. The sum, $w_\ell = \sum_{j=1}^{n} \lambda_j^\ell$, of the ℓ-th powers of the eigenvalues is known as the ℓ-th spectral moment and is equal to the number of closed walks of length ℓ in G (see, for example, [5]). It is folklore that $w_1 = 0$, w_2 is two times the number of edges of G, and w_3 is six times the number of triangles in G. By using generating functions to count a certain type of closed walk, we extend this list of equations to beyond what appears in the current literature.

Let C_i denote the i-cycle. For any graph H, let $[H]$ denote the number of subgraphs in G that are isomorphic to H. For 4-regular bipartite G, equations for w_ℓ up to $\ell = 6$ were given by Cvetković et al. [1]:

$$\begin{aligned} w_0 &= n, \qquad & w_4 &= 28n + 8[C_4], \\ w_2 &= 4n, \qquad & w_6 &= 232n + 144[C_4] + 12[C_6]. \end{aligned} \tag{1}$$

Stevanović et al. [10] added an inequality for $\ell = 8$:

$$w_8 \geqslant 2092n + 2024[C_4] + 288[C_6]. \tag{2}$$

We give a general method for finding equations such as (1) for spectral moments of regular graphs, whether or not they are bipartite. The subgraphs that appear in our equations will be called *contributors* (in (1) all of the contributors are cycles, but this is not true in general). Our methods build on those of Friedland et al. [3] and Wanless [11] who relate the numbers of matchings to the counts of contributors. We also use a special class of closed walks similar to the "primitive circuits" used by McKay [8] to count spanning trees. Using this restricted class of closed walks we will derive generating functions that give expressions for all closed walks in terms of the counts of contributors.

Our notation for contributors is as follows. We let $C_{i_1 \cdot i_2 \cdots i_h}$ denote cycles of length $i_1, i_2, \ldots, i_h$ sharing a single vertex, $C_{i_1 - i_2}$ an i_1-cycle joined to an i_2-cycle by an edge, $\Theta_{i_1, i_2, \ldots, i_h}$ two vertices joined by internally disjoint paths of lengths $i_1, i_2, \ldots, i_h$, and K_i the complete graph on i vertices. Examples of this notation appear in Fig. 1. If at any point we encounter contributors that cannot be described by our notation, we draw a picture of the subgraph like those in Fig. 1.

Formally, we define a contributor to be a connected graph with minimum degree at least 2. As we will show in Theorem 9, a contributor affects w_i if and only if its edges can be covered by a closed walk of length i. Any given w_i is only affected by finitely many contributors because a contributor with j edges can only affect w_i if $j \leqslant i$. We note that although our contributors are the same as in [11], they are used differently because we are counting slightly different types of walks.